Second-Order Linear Dynamic Systems

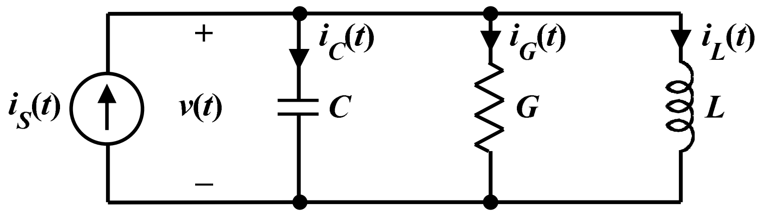
© R. Eugene Stuffle, Ph.D., M.S.B.A., P.E.

Second-order linear dynamic systems are described by equations of the form:

$$\frac{d^2y(t)}{dt^2}+2\zeta\omega_n\frac{dy}{dt}+\omega_n^2y(t)=A\omega_n^2z(t) \tag{1.1}$$

where $y(t)$ is the system response, or output, and $z(t)$ is the forcing function, or input. The symbols adopted here are a commonly used engineering notation, regardless of the field of concern. ζ is called the damping ratio, A is the *DC* or *static* gain, and ω_n is the natural frequency of the system. Several examples are given below.

Parallel RLC Circuit



Using the elementary component *i*-*v* relationships, we write:

$$v(t) = L\frac{di_L(t)}{dt} \tag{1.2}$$

$$i_G(t) = Gv(t) = G\left[L\frac{di_L(t)}{dt}\right] = GL\frac{di_L(t)}{dt} \tag{1.3}$$

$$i_C(t) = C\frac{dv(t)}{dt} = C\frac{d}{dt}\left[L\frac{di_L(t)}{dt}\right] = CL\frac{d^2i_L(t)}{dt^2} \tag{1.4}$$

Upon applying Kirchhoff's Current Law

$$i_C(t) + i_G(t) + i_L(t) = i_S(t) \tag{1.5}$$

we see that this circuit can be described by the second-order linear ordinary differential equation:

$$CL\frac{d^2i_L(t)}{dt^2} + GL\frac{di_L(t)}{dt} + i_L(t) = i_S(t) \tag{1.6}$$

or

$$\frac{d^2i_L(t)}{dt^2} + \frac{G}{C}\frac{di_L(t)}{dt} + \frac{1}{LC}i_L(t) = \frac{1}{LC}i_S(t) \tag{1.7}$$

Comparing this result to equation (1.1), we see that

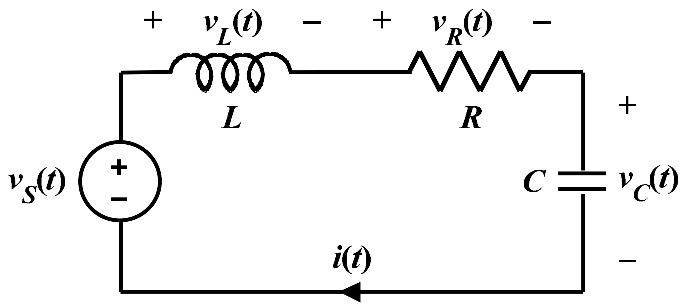
$$\omega_n^2 = \frac{1}{LC} \quad \Rightarrow \quad \omega_n = \frac{1}{\sqrt{LC}} \tag{1.8}$$

$$A\omega_n^2 = \frac{1}{LC} \quad \Rightarrow \quad A = 1 \tag{1.9}$$

and

$$2\zeta\omega_n = \frac{G}{C} \quad \Rightarrow \quad \zeta = \frac{G}{2\omega_n C} = \frac{G}{2}\sqrt{\frac{L}{C}} \tag{1.10}$$

Series RLC Circuit



Using the elementary component *i*-*v* relationships, we write:

$$i(t) = C\frac{dv_C(t)}{dt} \tag{1.11}$$

$$v_R(t) = Ri(t) = R\left[C\frac{dv_C(t)}{dt}\right] = RC\frac{dv_C(t)}{dt} \tag{1.12}$$

$$v_L(t) = L\frac{di(t)}{dt} = L\frac{d}{dt}\left[C\frac{dv_C(t)}{dt}\right] = LC\frac{d^2v_C(t)}{dt^2} \tag{1.13}$$

Upon applying Kirchhoff's Voltage Law

$$v_L(t) + v_R(t) + v_C(t) = v_S(t) \tag{1.14}$$

we see that this circuit can be described by the second-order linear ordinary differential equation:

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v_S(t) \tag{1.15}$$

or

$$\frac{d^2v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{1}{LC}v_S(t) \tag{1.16}$$

Comparing this result to equation (1.1), we see that

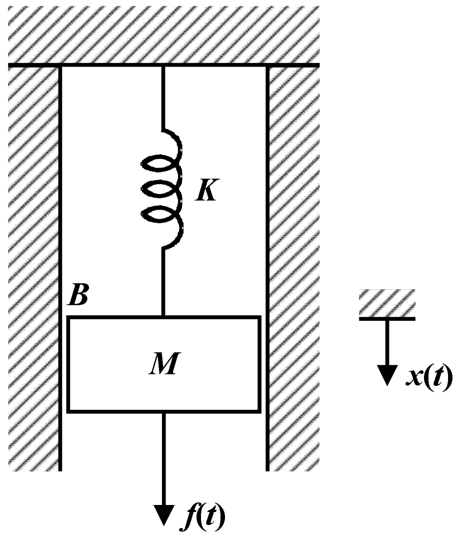
$$\omega_n^2 = \frac{1}{LC} \quad \Rightarrow \quad \omega_n = \frac{1}{\sqrt{LC}} \tag{1.17}$$

$$A\omega_n^2 = \frac{1}{LC} \quad \Rightarrow \quad A = 1 \tag{1.18}$$

and

$$2\zeta\omega_n = \frac{R}{L} \quad \Rightarrow \quad \zeta = \frac{R}{2\omega_n L} = \frac{R}{2}\sqrt{\frac{C}{L}} \tag{1.19}$$

Translational Mechanical System



When $f(t)$ is applied, friction and the spring will resist any motion so that, according to Newton's Second Law of Motion,

$$f(t) - Bv(t) - Kx(t) = Ma(t) \tag{1.20}$$

where $v(t) = \frac{dx(t)}{dt}$ is the velocity, and $a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left[\frac{dx(t)}{dt}\right] = \frac{d^2x(t)}{dt^2}$ is the acceleration.

Substituting these into equation (1.20) yields the second-order linear ordinary differential equation:

$$f(t) - B\frac{dx(t)}{dt} - Kx(t) = M\frac{d^2x(t)}{dt^2} \tag{1.21}$$

or

$$\frac{d^2x(t)}{dt^2} + \frac{B}{M}\frac{dx(t)}{dt} + \frac{K}{M}x(t) = \frac{1}{M}f(t) \tag{1.22}$$

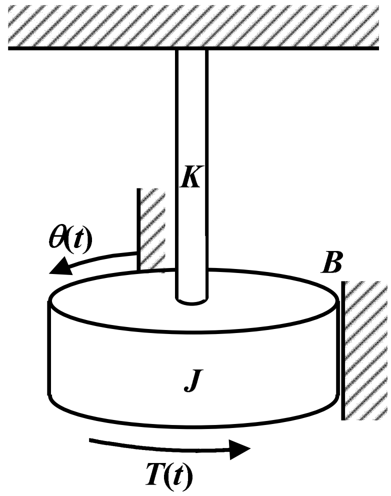
verifying that Newton's Second Law of Motion is clearly a mechanical equivalent to Kirchhoff's Laws for electrical circuits. Comparing this result to equation (1.1), we see that

$$\omega_n^2 = \frac{K}{M} \quad \Rightarrow \quad \omega_n = \sqrt{\frac{K}{M}} \tag{1.23}$$

$$A\omega_n^2 = \frac{1}{M} \quad \Rightarrow \quad A = \frac{1}{K} \tag{1.24}$$

and $$2\zeta\omega_n = \frac{B}{M} \quad \Rightarrow \quad \zeta = \frac{B}{2\omega_n M} = \frac{B}{2\sqrt{KM}} \tag{1.25}$$

Rotational Mechanical System



When $T(t)$ is applied, friction and the spring will resist any motion so that, according to Newton's Second Law of Motion,

$$T(t) - B\omega(t) - K\theta(t) = J\alpha(t) \tag{1.26}$$

where $\omega(t) = \dfrac{d\theta(t)}{dt}$ is the angular velocity, and $\alpha(t) = \dfrac{d\omega(t)}{dt} = \dfrac{d}{dt}\left[\dfrac{d\theta(t)}{dt}\right] = \dfrac{d^2\theta(t)}{dt^2}$ is the angular acceleration. Substituting these into equation (1.26) yields the second-order linear ordinary differential equation:

$$T(t) - B\frac{d\theta(t)}{dt} - K\theta(t) = J\frac{d^2\theta(t)}{dt^2} \tag{1.27}$$

or

$$\frac{d^2\theta(t)}{dt^2} + \frac{B}{J}\frac{d\theta(t)}{dt} + \frac{K}{J}\theta(t) = \frac{1}{J}T(t) \tag{1.28}$$

which again is clearly analogous to Kirchhoff's Laws for electrical circuits. Comparing this result to equation (1.1), we see that

$$\omega_n^2 = \frac{K}{J} \quad \Rightarrow \quad \omega_n = \sqrt{\frac{K}{J}} \tag{1.29}$$

$$A\omega_n^2 = \frac{1}{J} \quad \Rightarrow \quad A = \frac{1}{K} \tag{1.30}$$

and

$$2\zeta\omega_n = \frac{B}{J} \quad \Rightarrow \quad \zeta = \frac{B}{2\omega_n J} = \frac{B}{2\sqrt{KJ}} \tag{1.31}$$

All of the examples considered here yield equations that are of the form of equation (1.1). Note that, if $z(t)=0$, the differential equation is said to be homogeneous, and the system response under that condition is called the ***natural*** response. If $z(t)\neq 0$, the differential equation is said to be non-homogeneous, and the ***complete*** response of the system with the forcing function applied is a combination of the ***natural*** response and additional term(s) called the ***forced*** response. Sometimes, these are called, respectively, the ***complimentary*** response and the ***particular*** response.

Zero-Input (Unforced) Systems

Consider the zero-input (homogeneous) form of equation (1.1):

$$\frac{d^2y(t)}{dt^2}+2\zeta\omega_n\frac{dy(t)}{dt}+\omega_n^2y(t)=0 \tag{1.32}$$

If we assume that the natural response of the system is exponential, i.e., $y(t)=\beta e^{rt}$, then

$$r^2\beta e^{rt}+2\zeta\omega_n r\beta e^{rt}+\omega_n^2\beta e^{rt}=0 \tag{1.33}$$

or

$$\left(r^2+2\zeta\omega_n r+\omega_n^2\right)\beta e^{rt}=0 \tag{1.34}$$

which means that

$$r^2+2\zeta\omega_n r+\omega_n^2=0 \tag{1.35}$$

Equation (1.35) is called the characteristic equation of the system, and it has roots given by:

$$\begin{aligned} r_{1,2}&=\frac{-2\zeta\omega_n\pm\sqrt{(2\zeta\omega_n)^2-4\omega_n^2}}{2}\\ &=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}\\ &=\left(-\zeta\pm\sqrt{\zeta^2-1}\right)\omega_n \end{aligned} \tag{1.36}$$

From this, we will see that there are four distinctly different forms of the solution to equation (1.32), depending on the value of ζ with respect to the number 1.

Case 1

If $\zeta > 1$, then $\zeta^2 - 1 > 0$, and there will be two distinct negative real roots, $r_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$ and $r_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$. In this case, the system is said to be ***overdamped***, and because there are ***two*** roots to the characteristic equation, $y(t)$ will have ***two*** exponential components:

$$y(t) = \beta_1 e^{r_1 t} + \beta_2 e^{r_2 t} \tag{1.37}$$

To determine the values of β_1 and β_2 note that

$$\dot{y}(t) = \beta_1 r_1 e^{r_1 t} + \beta_2 r_2 e^{r_1 t} \tag{1.38}$$

Evaluating equations (1.37) and (1.38) at $t = 0$, we have

$$\beta_1 + \beta_2 = y(0) \tag{1.39}$$

and

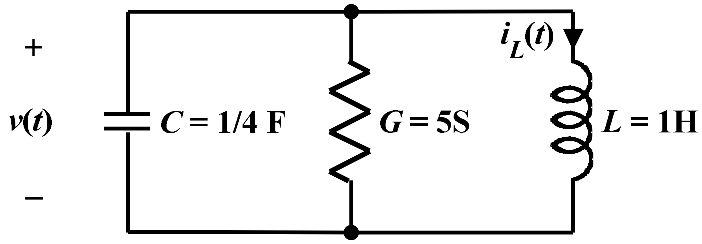
$$r_1 \beta_1 + r_2 \beta_2 = \dot{y}(0) \tag{1.40}$$

These two simultaneous equations can be used to evaluate β_1 and β_2 using Cramer's Rule as follows:

$$\beta_1 = \frac{\begin{vmatrix} y(0) & 1 \\ \dot{y}(0) & r_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix}} = \frac{r_2 y(0) - \dot{y}(0)}{r_2 - r_1} \tag{1.41}$$

$$\beta_2 = \frac{\begin{vmatrix} 1 & y(0) \\ r_1 & \dot{y}(0) \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix}} = \frac{\dot{y}(0) - r_1 y(0)}{r_2 - r_1} \tag{1.42}$$

Example 1.1



As shown by equation (1.7), this parallel RLC circuit can be described by the equation

$$\frac{d^2 i_L}{dt^2} + 20\frac{di_L}{dt} + 4i_L = 0 \tag{1.43}$$

Hence, the characteristic equation is

$$r^2 + 20r + 4 = 0 \tag{1.44}$$

and

$$\omega_n = \frac{1}{\sqrt{1(1/4)}} = 2 \tag{1.45}$$

$$\zeta = \frac{5}{2}\sqrt{\frac{1}{(1/4)}} = 5 \tag{1.46}$$

This is an overdamped system, with

$$r_1 = \left(-5 + \sqrt{25-1}\right)2 \approx -0.202 \tag{1.47}$$

$$r_2 = \left(-5 - \sqrt{25-1}\right)2 \approx -19.798 \tag{1.48}$$

Suppose now that $i_L(0) = 0$ and $v(0) = 1$. Then, $v(t) = L\frac{di_L(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{di_L}{dt}\right|_{t=0} = \frac{1}{L}v(0) = 1$, and

$$\beta_1 \approx \frac{(-19.798)(0) - 1}{-19.798 - (-0.202)} \approx \frac{-1}{-19.596} \approx 0.051 \tag{1.49}$$

$$\beta_2 \approx \frac{1 - (-0.202)(0)}{-19.798 - (-0.202)} \approx \frac{1}{-19.596} \approx -0.051 \tag{1.50}$$

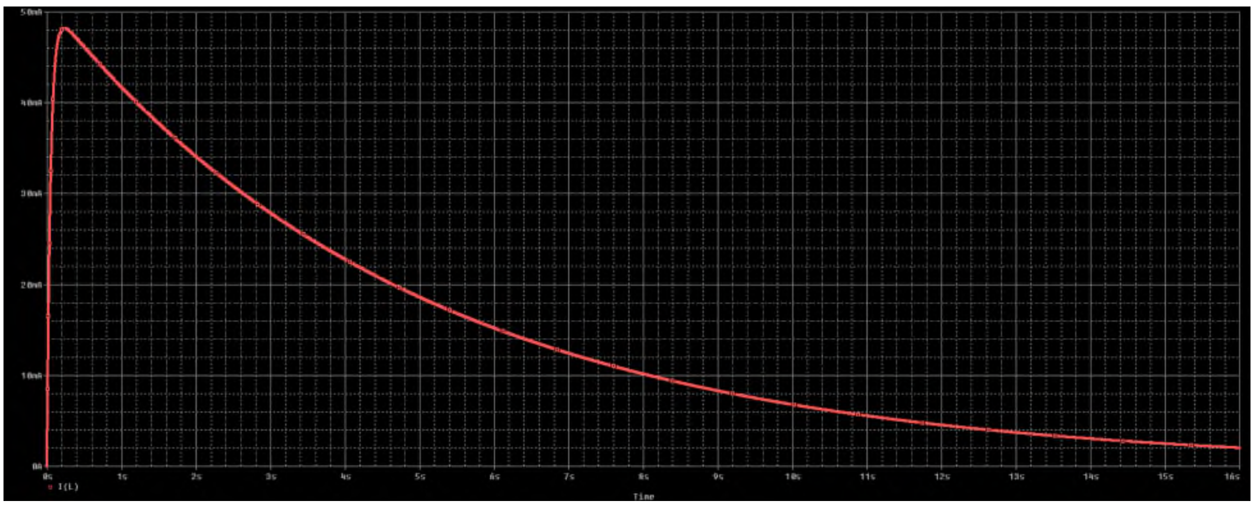
Hence,

$$i_L(t) \approx 0.051e^{-0.202t} - 0.051e^{-19.798t} \text{ A for } t > 0 \tag{1.51}$$

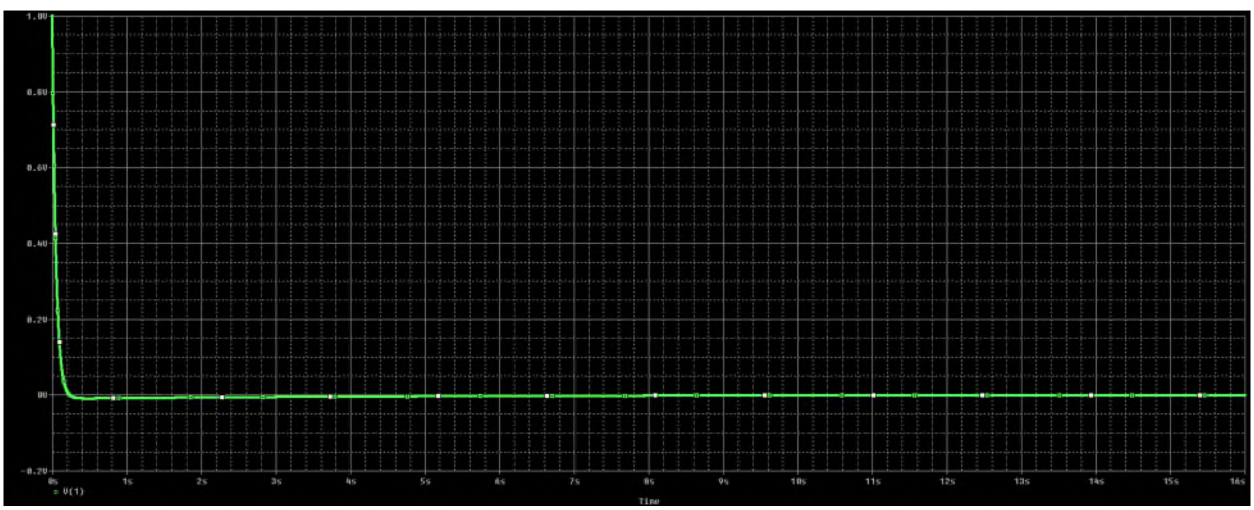
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 1.1
C     1    0    {1/4}      IC=1
G     1    0    1    0     5
L     1    0    1    IC=0
.TRAN      1    16   0     1m    UIC
.PROBE
.END
```

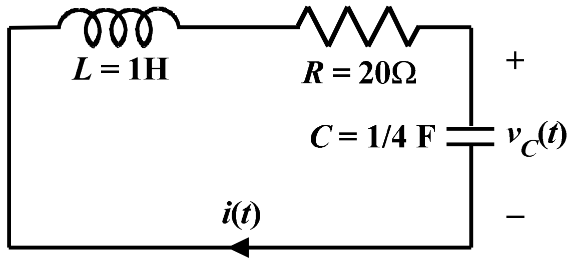
The inductor current is:



and the capacitor voltage is:



Example 1.2



As shown by equation (1.16), this series RLC circuit can be described by the equation

$$\frac{d^2 v_C}{dt^2} + 20\frac{dv_C}{dt} + 4v_C = 0 \tag{1.52}$$

Hence, the characteristic equation is

$$r^2 + 20r + 4 = 0 \tag{1.53}$$

and

$$\omega_n = \frac{1}{\sqrt{1(1/4)}} = 2 \tag{1.54}$$

$$\zeta = \frac{20}{2}\sqrt{\frac{(1/4)}{1}} = 5 \tag{1.55}$$

This is an overdamped system, with

$$r_1 = \left(-5 + \sqrt{25-1}\right)2 \approx -0.202 \tag{1.56}$$

$$r_2 = \left(-5 - \sqrt{25-1}\right)2 \approx -19.798 \tag{1.57}$$

Suppose now that $v_C(0) = 0$ and $i(0) = 1$. Then, $i(t) = C\frac{dv_C(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{dv_C(t)}{dt}\right|_{t=0} = \frac{1}{C}i(0) = 4$, and

$$\beta_1 \approx \frac{(-19.798)(0) - 4}{-19.798 - (-0.202)} \approx 0.204 \tag{1.58}$$

$$\beta_2 \approx \frac{4 - (-0.202)(0)}{-19.798 - (-0.202)} \approx -0.204 \tag{1.59}$$

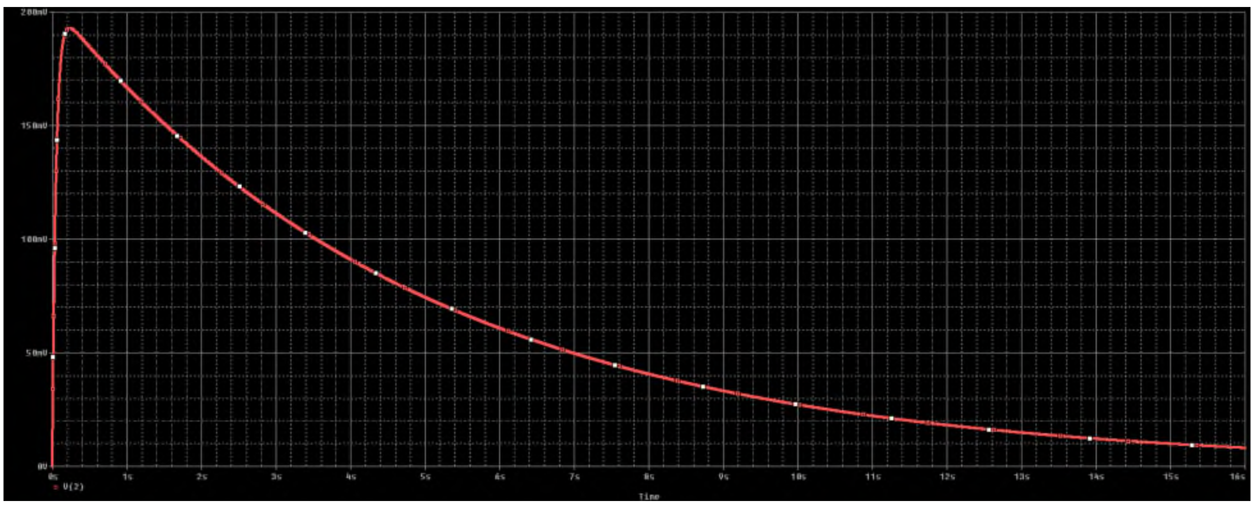
Hence,

$$v_C\left(t\right) \approx 0.204e^{-0.202t} - 0.204e^{-19.798t} \text{ V for } t > 0 \qquad (1.60)$$

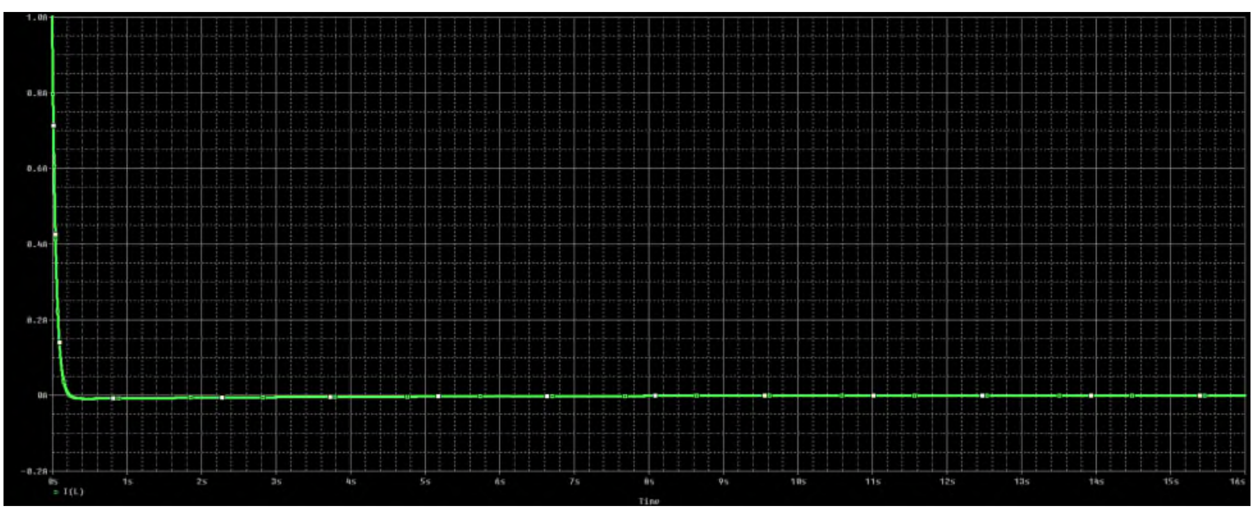
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 1.2
L     0    1    1    IC=1
R     1    2    20
C     2    0    {1/4}      IC=0
.TRAN      1    16   0     1m    UIC
.PROBE
.END
```

The capacitor voltage is:



and the inductor current is:



Case 2

If $0 < \zeta < 1$, then $\zeta^2 - 1 < 0$, and there will be two complex conjugate roots, $r_1 = \left(-\zeta + j\sqrt{1-\zeta^2}\right)\omega_n = -\zeta\omega_n + j\omega_d$ and $r_2 = \left(-\zeta - j\sqrt{1-\zeta^2}\right)\omega_n = -\zeta\omega_n - j\omega_d$. In this case, the system is said to be ***underdamped***., and the quantity $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is called the *damped* or *ringing* frequency.

As in Case 1, because there are ***two*** distinct roots to the characteristic equation, $y(t)$ has ***two*** exponential components:

$$\begin{aligned} y(t) &= \beta_1 e^{(-\zeta\omega_n + j\omega_d)t} + \beta_2 e^{(-\zeta\omega_n - j\omega_d)t} \\ &= e^{-\zeta\omega_n t}\left(\beta_1 e^{j\omega_d t} + \beta_2 e^{-j\omega_d t}\right) \end{aligned} \tag{1.61}$$

However, it is usually preferred to use Euler's identity

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta \tag{1.62}$$

to express $y(t)$ in the alternate form

$$\begin{aligned} y(t) &= e^{-\zeta\omega_n t}\left[\beta_1\left(\cos\omega_d t + j\sin\omega_d t\right) + \beta_2\left(\cos\omega_d t - j\sin\omega_d t\right)\right] \\ &= e^{-\zeta\omega_n t}\left[\left(\beta_1 + \beta_2\right)\cos\omega_d t + j\left(\beta_1 - \beta_2\right)\sin\omega_d t\right] \\ &= e^{-\zeta\omega_n t}\left[B_1\cos\omega_d t + B_2\sin\omega_d t\right] \end{aligned} \tag{1.63}$$

where $B_1 = \beta_1 + \beta_2$ and $B_2 = j\left(\beta_1 - \beta_2\right)$.

To determine the values of B_1 and B_2 note that

$$\dot{y}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}\left[B_1\cos\omega_d t + B_2\sin\omega_d t\right] + e^{-\zeta\omega_n t}\left[-B_1\omega_d\sin\omega_d t + B_2\omega_d\cos\omega_d t\right] \tag{1.64}$$

Evaluating equations (1.63) and (1.64) at $t = 0$, we have

$$B_1 = y(0) \tag{1.65}$$

and

$$-\zeta\omega_n B_1 + B_2\omega_d = \dot{y}(0) \tag{1.66}$$

Thus,

$$B_2 = \frac{\dot{y}(0) + \zeta\omega_n B_1}{\omega_d} = \frac{\dot{y}(0) + \zeta\omega_n y(0)}{\omega_d} \tag{1.67}$$

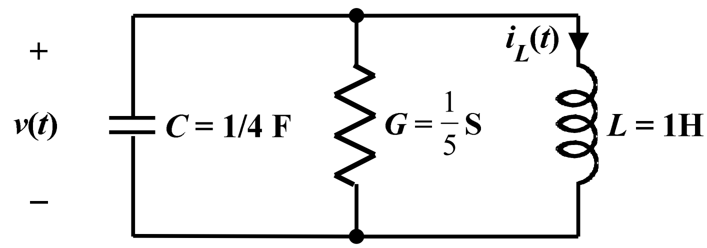
Alternately, note that

$$B_1 \cos\omega_d t + B_2 \sin\omega_d t = B_3 \cos(\omega_d t - \phi) \tag{1.68}$$

where $B_3 = \sqrt{B_1^2 + B_2^2}$ and $\phi = \tan^{-1}\left(\frac{B_2}{B_1}\right)$, so that $y(t)$ can be written in a slightly more compact form as

$$y(t) = B_3 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi) \tag{1.69}$$

Example 2.1



As shown by equation (1.7), this parallel RLC circuit can be described by the equation

$$\frac{d^2 i_L}{dt^2} + \frac{4}{5}\frac{di_L}{dt} + 4i_L = 0 \tag{1.70}$$

Hence, the characteristic equation is

$$r^2 + \frac{4}{5}r + 4 = 0 \tag{1.71}$$

and

$$\omega_n = 2 \tag{1.72}$$

$$\zeta = \frac{1}{10}\sqrt{4} = 0.2 \tag{1.73}$$

$$\omega_d = 2\sqrt{1-(0.2)^2} \approx 1.960 \tag{1.74}$$

This is an underdamped system, with

$$r_{1,2} \approx -0.400 + j1.960 \tag{1.75}$$

$$r_{1,2} \approx -0.400 - j1.960 \tag{1.76}$$

Suppose now that $i_L(0) = 0$ and $v(0) = 1$. Then, $v(t) = L\frac{di_L(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{di_L(t)}{dt}\right|_{t=0} = \frac{1}{L}v(0) = 1$, and

$$B_1 = 0 \tag{1.77}$$

$$B_2 \approx \frac{1+(0.2)(2)(0)}{1.960} \approx 0.510 \tag{1.78}$$

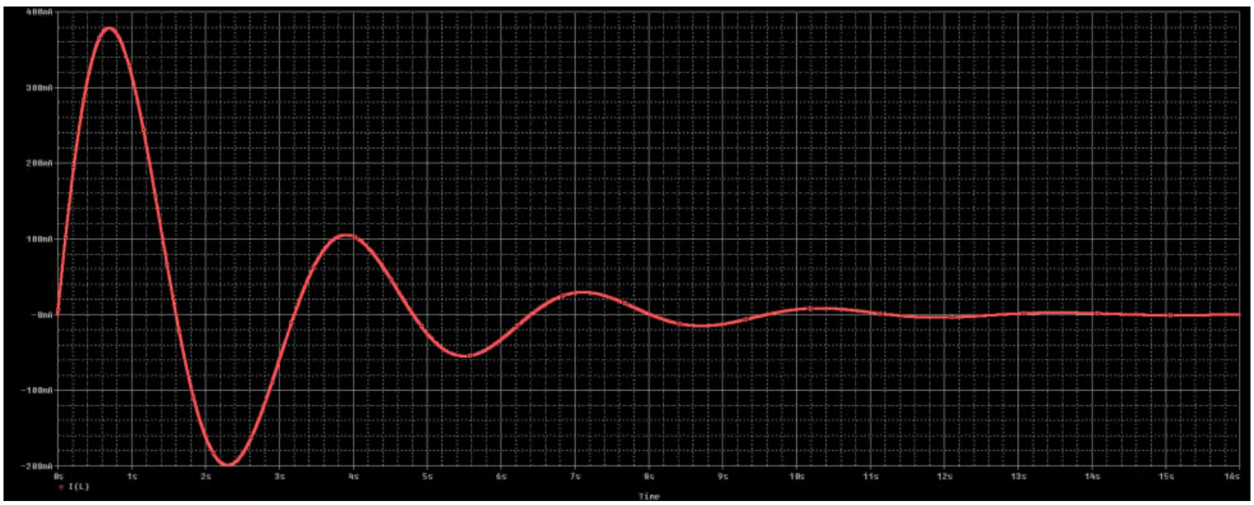
Hence,

$$i_L(t) \approx 0.510e^{-0.4t}\sin(1.960t) \text{ A for } t>0 \tag{1.79}$$

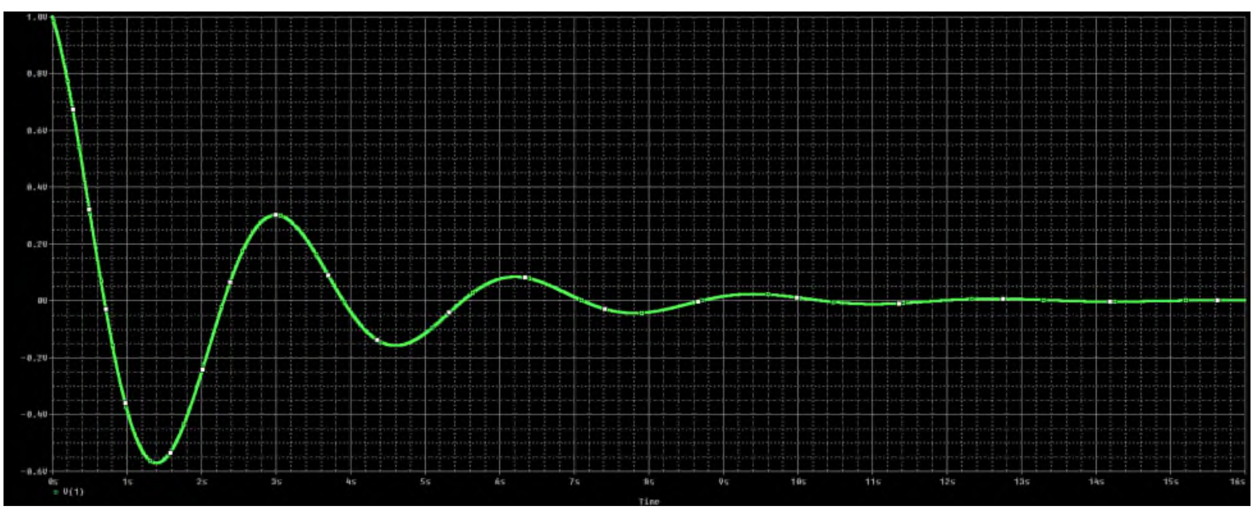
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 2.1
C     1     0     {1/4}       IC=1
G     1     0     1     0     {1/5}
L     1     0     1     IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
```

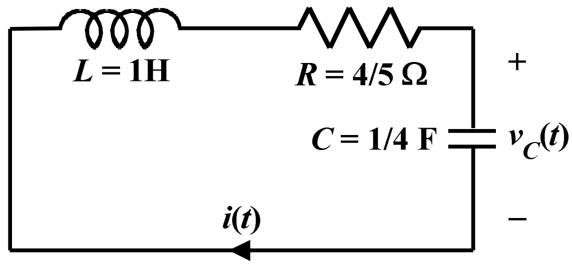
The inductor current is:



and the capacitor voltage is:



Example 2.2



As shown by equation (1.16), this series RLC circuit can be described by the equation

$$\frac{d^2 v_C}{dt^2} + \frac{4}{5}\frac{dv_C}{dt} + 4v_C = 0 \tag{1.80}$$

Hence, the characteristic equation is

$$r^2 + \frac{4}{5}r + 4 = 0 \tag{1.81}$$

and

$$\omega_n = 2 \tag{1.82}$$

$$\zeta = \frac{2}{5}\sqrt{\frac{1}{4}} = 0.2 \tag{1.83}$$

$$\omega_d = 2\sqrt{1-(0.2)^2} \approx 1.960 \tag{1.84}$$

This is an underdamped system, with

$$r_1 = -0.400 + j1.960 \tag{1.85}$$

$$r_2 = -0.400 - j1.960 \tag{1.86}$$

Suppose now that $v_C(0) = 0$ and $i(0) = 1$. Then, $i(t) = C\frac{dv_C(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{dv_C(t)}{dt}\right|_{t=0} = \frac{1}{C}i(0) = 4$, and

$$B_1 = 0 \tag{1.87}$$

$$B_2 \approx \frac{4+(0.2)(2)(0)}{1.960} \approx 2.041 \tag{1.88}$$

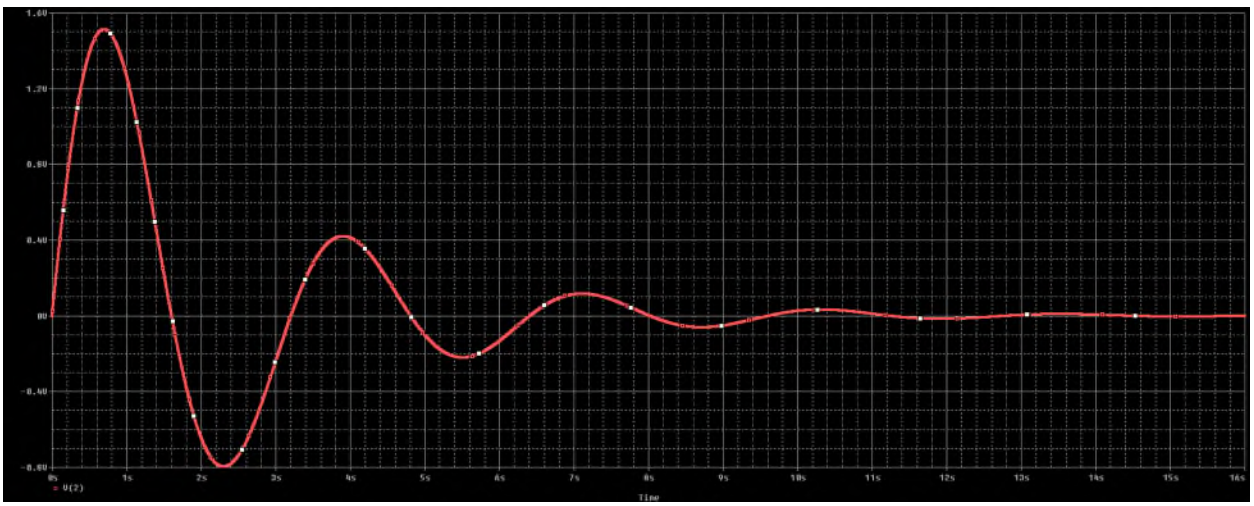
Hence,

$$v_C(t) = 2.041e^{-0.4t}\sin(1.960t) \quad \text{V} \quad t>0 \tag{1.89}$$

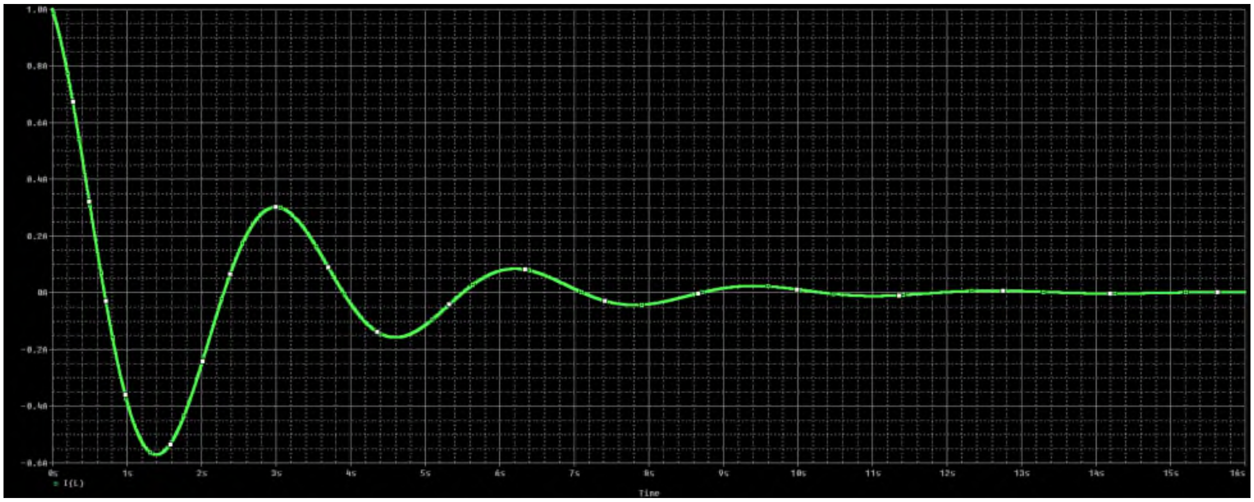
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 2.2
L    0    1    1    IC=1
R    1    2    {4/5}
C    2    0    {1/4}     IC=0
.TRAN     1    16   0    1m   UIC
.PROBE
.END
```

The capacitor voltage is:



and the inductor current is:


1.0A
0.8A
0.6A
0.4A
0.2A
0A
-0.2A
-0.4A
-0.6A
0s
1s
2s
3s
4s
5s
6s
7s
8s
9s
10s
11s
12s
13s
14s
15s
16s
I(L)
Time

Case 3

If $\zeta = 1$, then $\zeta^2 - 1 = 0$, and there will be two identical negative real roots, $r_1 = r_2 = -\omega_n$. In this case, the system is said to be ***critically damped***. This case can be considered to be the “borderline” between overdamped and underdamped systems.

The general form of the solution is

$$y(t) = (\beta_1 + \beta_2 t)e^{-\omega_n t} \tag{1.90}$$

To determine the values of β_1 and β_2 note that

$$\dot{y}(t) = \beta_2 e^{-\omega_n t} - \omega_n(\beta_1 + \beta_2 t)e^{-\omega_n t} \tag{1.91}$$

Evaluating equations (1.90) and (1.91) at $t = 0$, we have

$$\beta_1 = y(0) \tag{1.92}$$

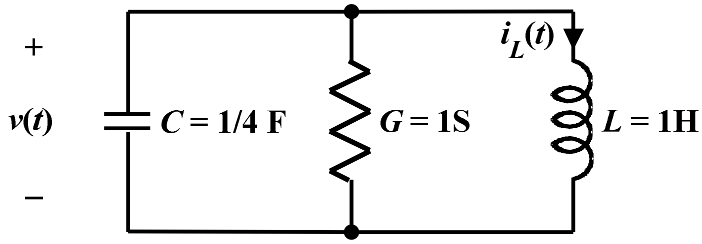
and

$$\beta_2 - \omega_n \beta_1 = \dot{y}(0) \tag{1.93}$$

Thus,

$$\beta_2 = \dot{y}(0) + \omega_n \beta_1 = \dot{y}(0) + \omega_n y(0) \tag{1.94}$$

Example 3.1



As shown by equation (1.7), this parallel circuit can be described by the equation

$$\frac{d^2 i_L}{dt^2} + 4\frac{di_L}{dt} + 4i_L = 0 \qquad (1.95)$$

Hence, the characteristic equation is

$$r^2 + 4r + 4 = 0 \qquad (1.96)$$

and

$$\omega_n = 2 \qquad (1.97)$$

$$\zeta = \frac{1}{2}\sqrt{4} = 1 \qquad (1.98)$$

This is a critically damped system, with

$$r_1 = r_2 = -2 \qquad (1.99)$$

Suppose now that $i_L(0) = 0$ and $v(0) = 1$. Then, $v(t) = L\frac{di_L(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{di_L}{dt}\right|_{t=0} = \frac{1}{L}v(0) = 1$, and

$$\beta_1 = 0 \qquad (1.100)$$

$$\beta_2 = 1 + (2)(0) = 1 \qquad (1.101)$$

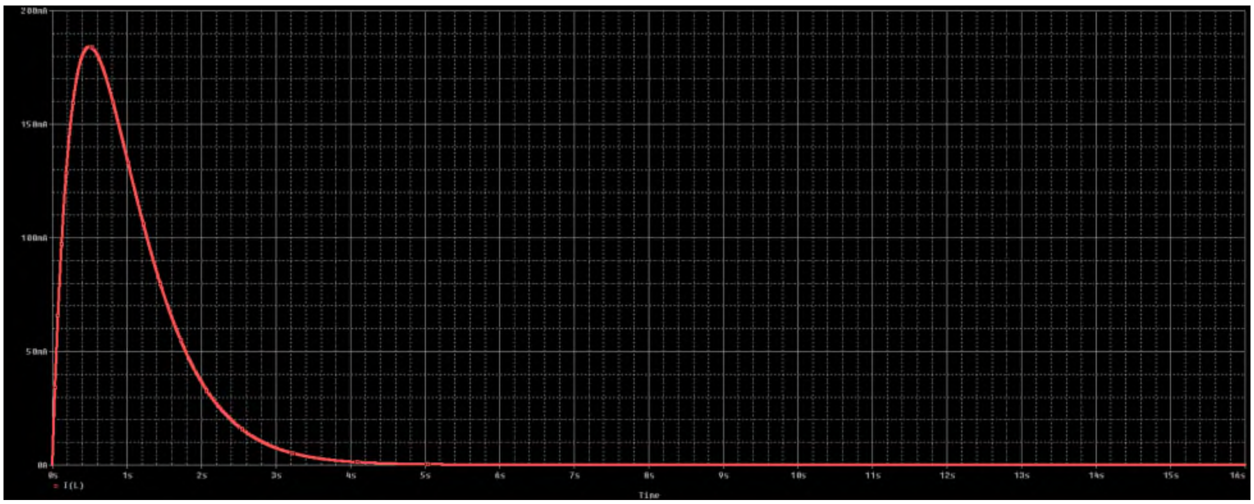
Hence,

$$i_L(t) = te^{-2t} \text{ A for } t > 0 \qquad (1.102)$$

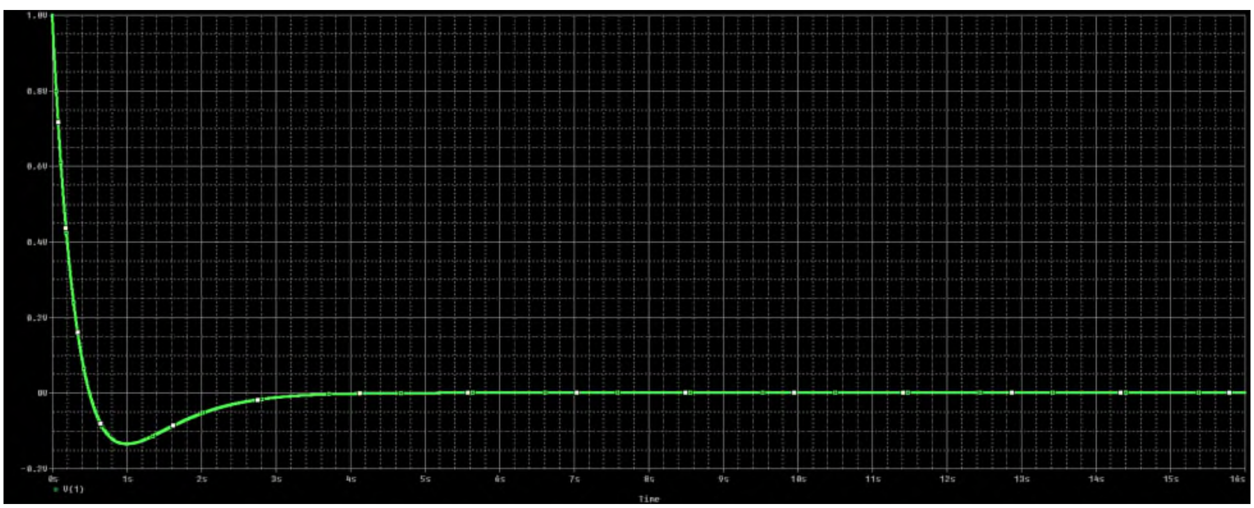
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 3.1
C     1     0     {1/4}      IC=1
G     1     0     1     0     1
L     1     0     1     IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
```

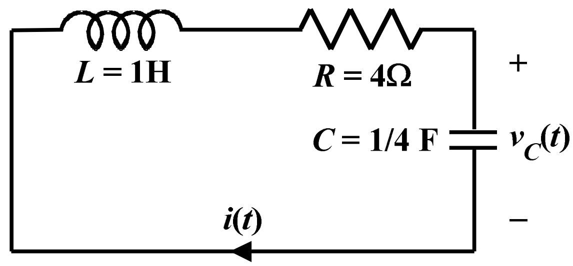
The inductor current is:



and the capacitor voltage is:



Example 3.2



As shown by equation (1.16), this series RLC circuit can be described by the equation

$$\frac{d^2 v_C}{dt^2} + 4\frac{dv_C}{dt} + 4v_C = 0 \tag{1.103}$$

Hence, the characteristic equation is

$$r^2 + 4r + 4 = 0 \tag{1.104}$$

and

$$\omega_n = 2 \tag{1.105}$$

$$\zeta = \frac{4}{2}\sqrt{\frac{1}{4}} = 1 \tag{1.106}$$

This is a critically damped system, with

$$r_1 = r_2 = -2 \tag{1.107}$$

Suppose now that $v_C(0) = 0$ and $i(0) = 1$. Then, $i(t) = C\frac{dv_C(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{dv_C(t)}{dt}\right|_{t=0} = \frac{1}{C}i(0) = 4$, and

$$\beta_1 = 0 \tag{1.108}$$

$$\beta_2 = 4 + (2)(0) = 4 \tag{1.109}$$

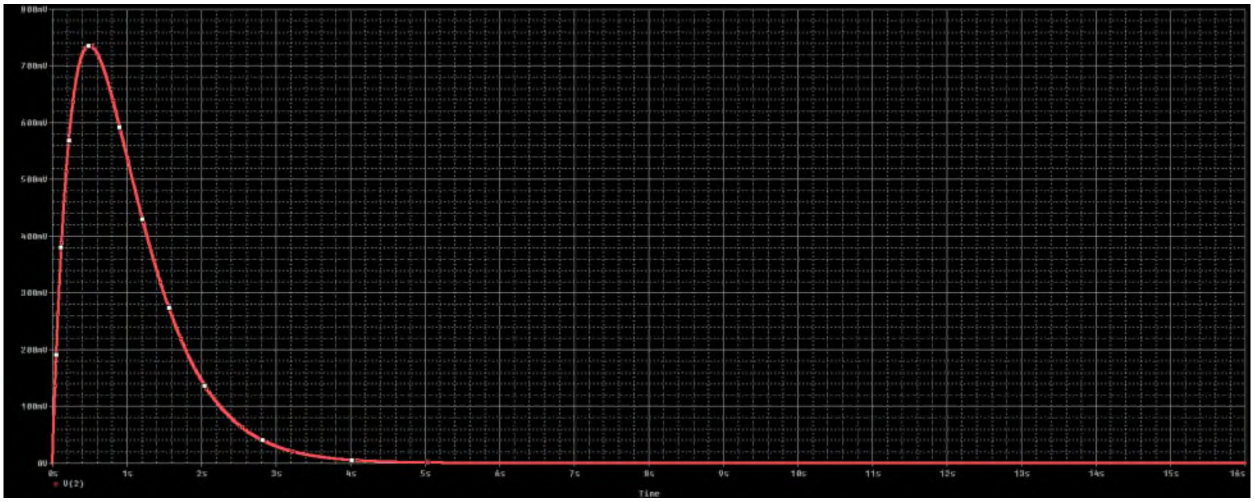
Hence,

$$v_C(t) = 4te^{-2t} \text{ V} \quad t > 0 \tag{1.110}$$

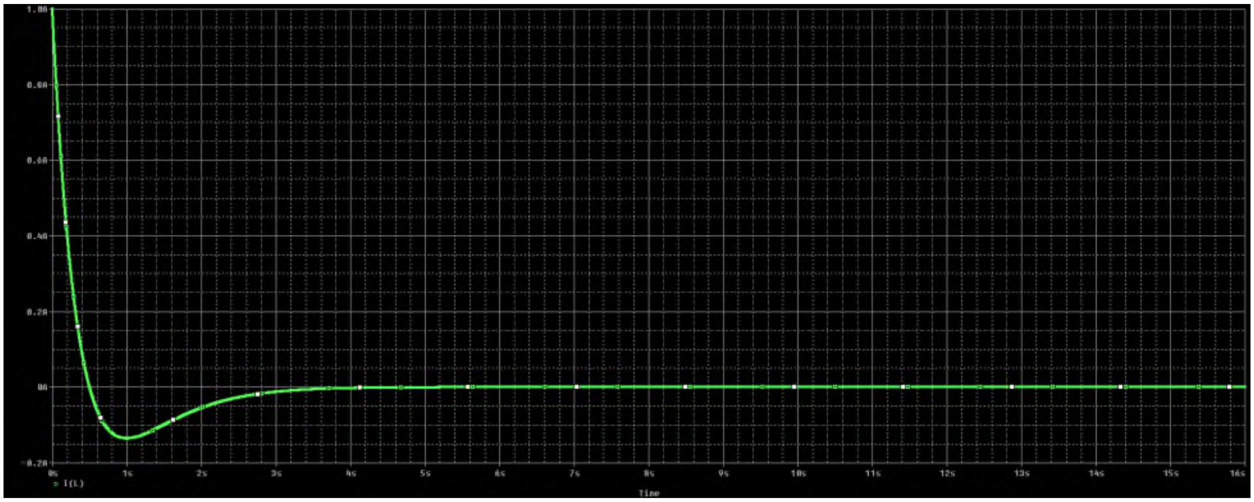
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 3.2
L     0    1    1    IC=1
R     1    2    4
C     2    0    {1/4}     IC=0
.TRAN      1    16   0    1m   UIC
.PROBE
.END
```

The capacitor voltage is:



and the inductor current is:



Case 4

If $\zeta = 0$, then $\zeta^2 - 1 = -1$, and there will be two conjugate imaginary roots, $r_{1,2} = \pm j\omega_n$. In this case, the system is said to be ***undamped***.

As there are ***two*** distinct roots to the characteristic equation, $y(t)$ has ***two*** exponential components

$$y(t) = \beta_1 e^{j\omega_n t} + \beta_2 e^{-j\omega_n t} \tag{1.111}$$

Here again, as in Case 2, it is usually preferred to use Euler's identity to express $y(t)$ in the alternate form

$$\begin{aligned} y(t) &= \beta_1 (\cos\omega_n t + j\sin\omega_n t) + \beta_2 (\cos\omega_n t - j\sin\omega_n t) \\ &= (\beta_1 + \beta_2)\cos\omega_n t + j(\beta_1 - \beta_2)\sin\omega_n t \\ &= B_1 \cos\omega_n t + B_2 \sin\omega_n t \end{aligned} \tag{1.112}$$

where $B_1 = \beta_1 + \beta_2$ and $B_2 = j(B_1 - B_2)$.

To determine the values of B_1 and B_2 note that

$$\dot{y}(t) = -\omega_n B_1 \sin\omega_n t + \omega_n B_2 \cos\omega_n t \tag{1.113}$$

Evaluating equations (1.112) and (1.113) at $t = 0$, we have

$$B_1 = y(0) \tag{1.114}$$

and

$$\omega_n B_2 = \dot{y}(0) \tag{1.115}$$

so that

$$B_2 = \frac{\dot{y}(0)}{\omega_n} \tag{1.116}$$

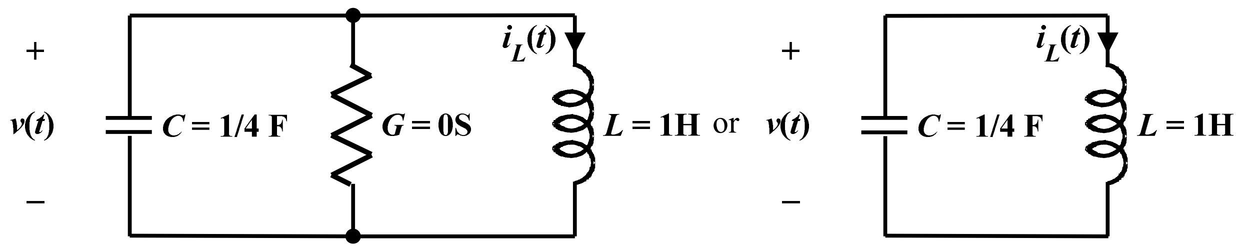
Alternately, note that

$$B_1 \cos\omega_n t + B_2 \sin\omega_n t = B_3 \cos(\omega_n t - \phi) \tag{1.117}$$

where $B_3 = \sqrt{B_1^2 + B_2^2}$ and $\phi = \tan^{-1}\left(\frac{B_2}{B_1}\right)$, so that $y(t)$ can be written in a slightly more compact form as

$$y(t) = B_3 \cos(\omega_n t - \phi) \tag{1.118}$$

Example 4.1



As shown by equation (1.7), this parallel RLC circuit can be described by the equation

$$\frac{d^2 i_L}{dt^2} + 4 i_L = 0 \qquad (1.119)$$

Hence, the characteristic equation is

$$r^2 + 4 = 0 \qquad (1.120)$$

and

$$\omega_n = 2 \qquad (1.121)$$

$$\zeta = 0 \qquad (1.122)$$

This is an undamped system, with

$$r_1 = j2 \qquad (1.123)$$

$$r_2 = -j2 \qquad (1.124)$$

Suppose now that $i_L(0) = 0$ and $v(0) = 1$. Then, $v(t) = L\frac{di_L(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{di_L}{dt}\right|_{t=0} = \frac{1}{L}v(0) = 1$, and

$$B_1 = 0 \qquad (1.125)$$

$$B_2 = \frac{1}{2} \qquad (1.126)$$

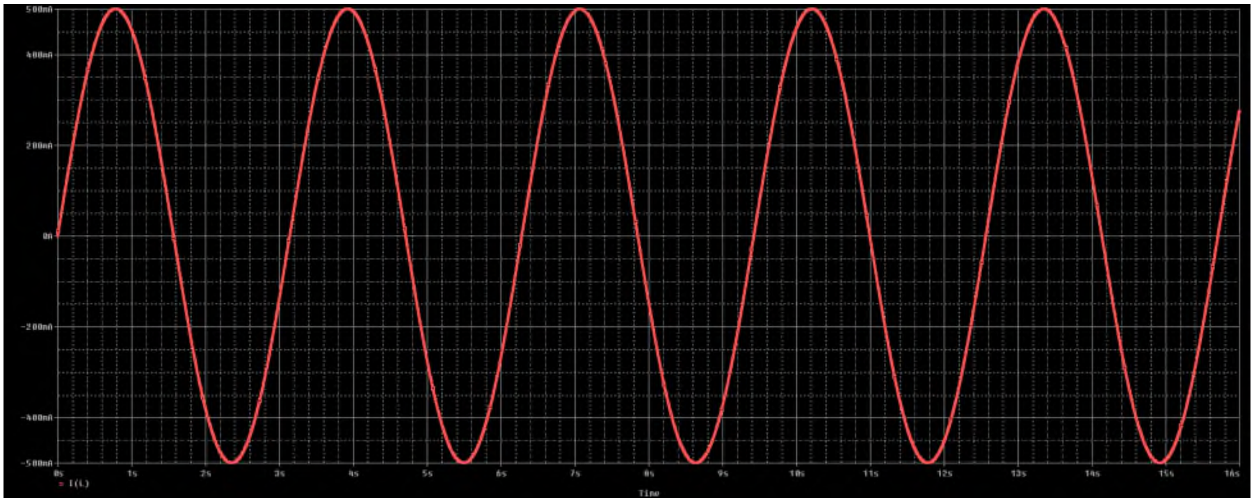
Hence,

$$i_L(t) = \frac{1}{2}\sin 2t \text{ A for } t > 0 \qquad (1.127)$$

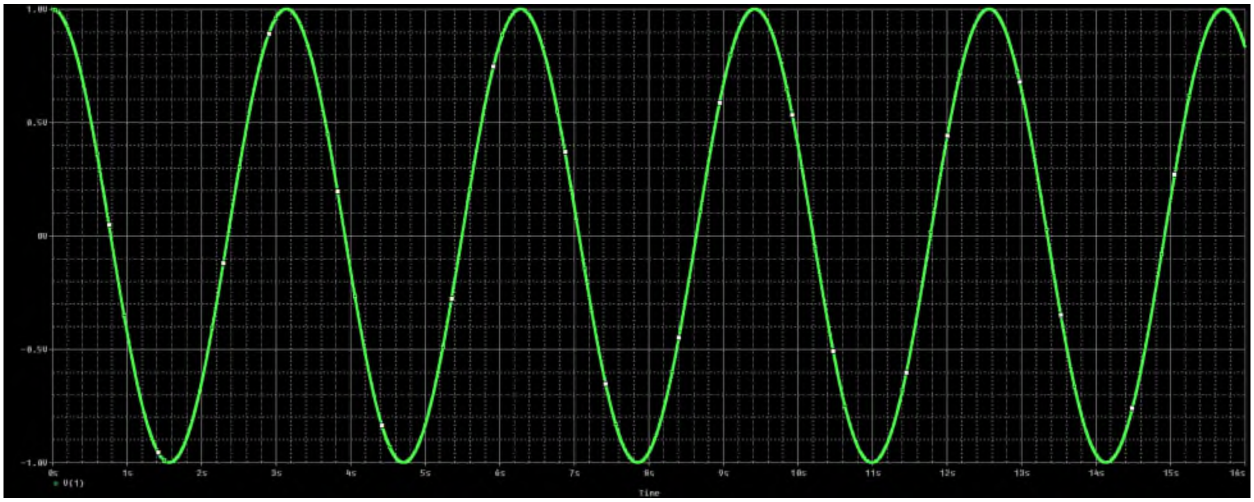
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 4.1
C    1    0    {1/4}     IC=1
L    1    0    1    IC=0
.TRAN     1    16   0    1m   UIC
.PROBE
.END
```

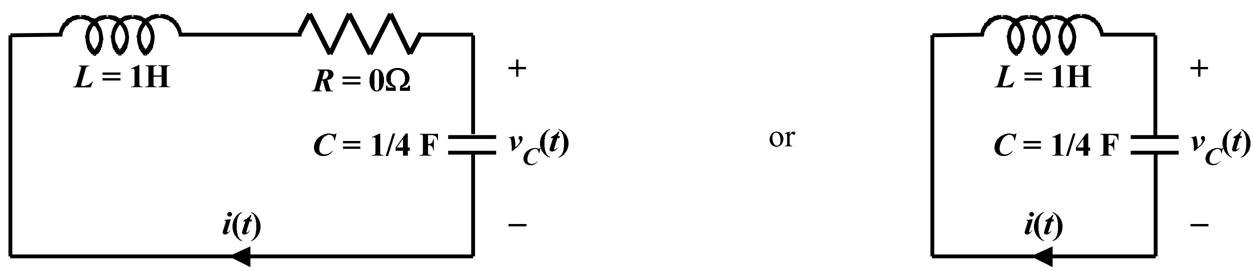
The inductor current is:



and the capacitor voltage is:



Example 4.2



As shown by equation (1.16), this series RLC circuit can be described by the equation

$$\frac{d^2 v_C}{dt^2} + 4v_C = 0 \tag{1.128}$$

Hence, the characteristic equation is

$$r^2 + 4 = 0 \tag{1.129}$$

and

$$\omega_n = 2 \tag{1.130}$$

$$\zeta = 0 \tag{1.131}$$

This is an undamped system, with

$$r_1 = j2 \tag{1.132}$$

$$r_2 = -j2 \tag{1.133}$$

Suppose now that $v_C(0) = 0$ and $i(0) = 1$. Then, $i(t) = C\frac{dv_C(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{dv_C(t)}{dt}\right|_{t=0} = \frac{1}{C} i(0) = 4$, and

$$B_1 = 0 \tag{1.134}$$

$$B_2 = \frac{4}{2} = 2 \tag{1.135}$$

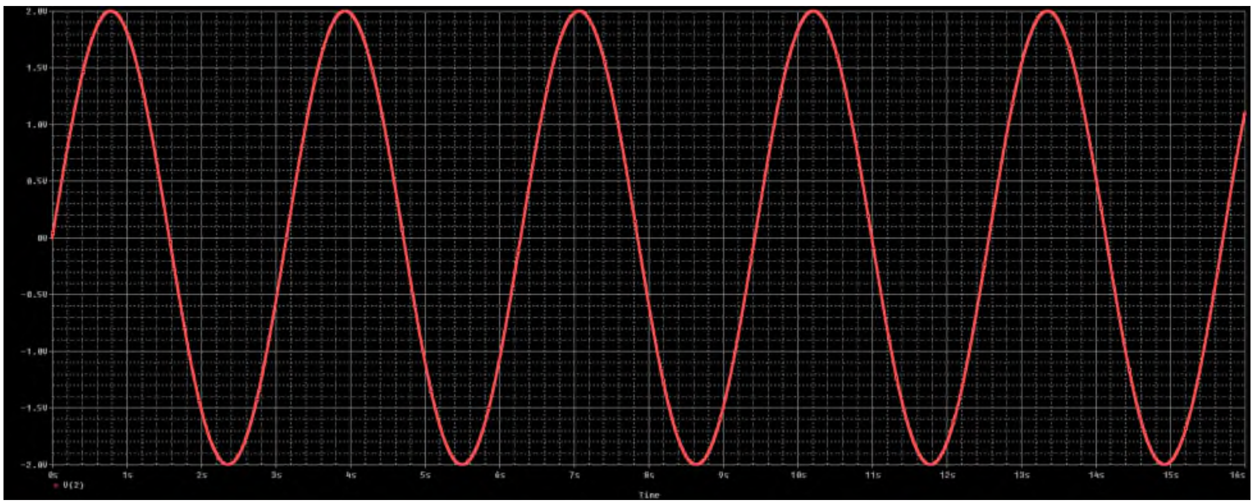
Hence,

$$v_C(t) = 2\sin 2t \quad \text{V} \quad t > 0 \tag{1.136}$$

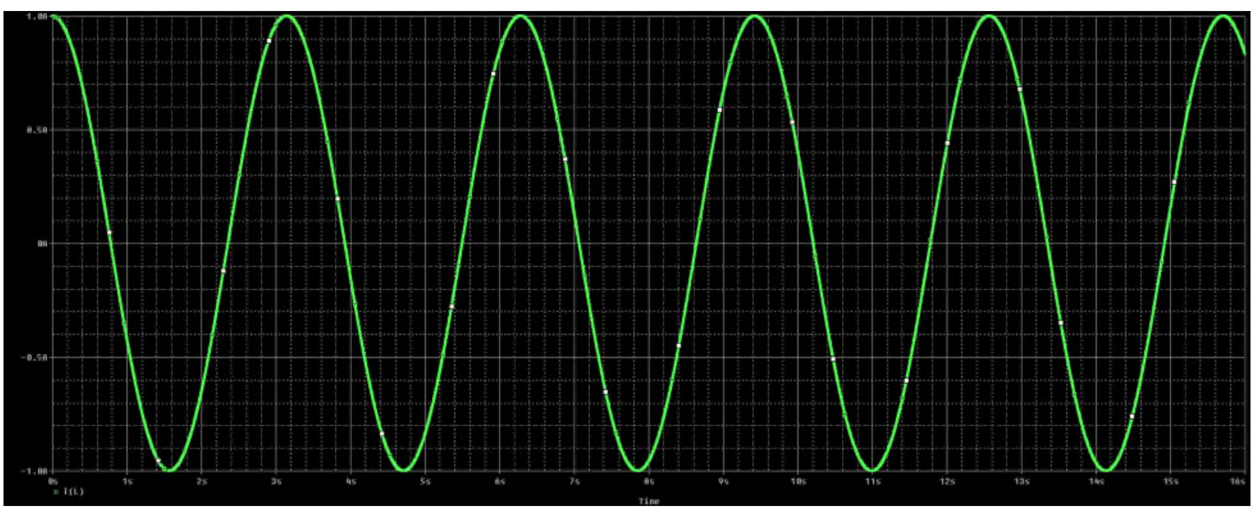
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 4.2
L     0     2     1     IC=1
C     2     0     {1/4}      IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
```

The capacitor voltage is:

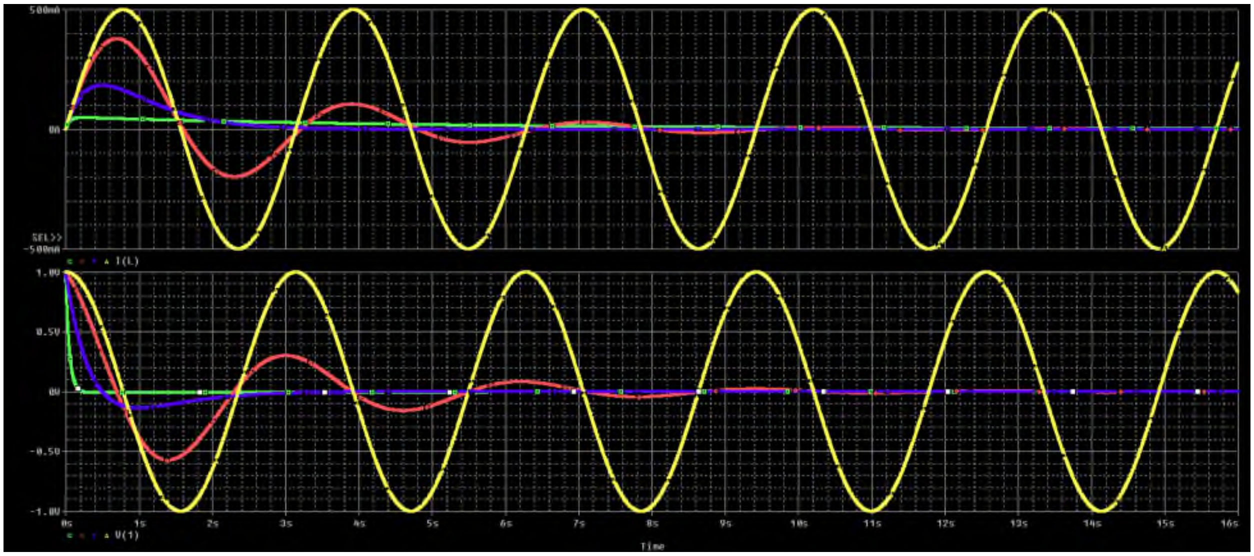


and the inductor current is:



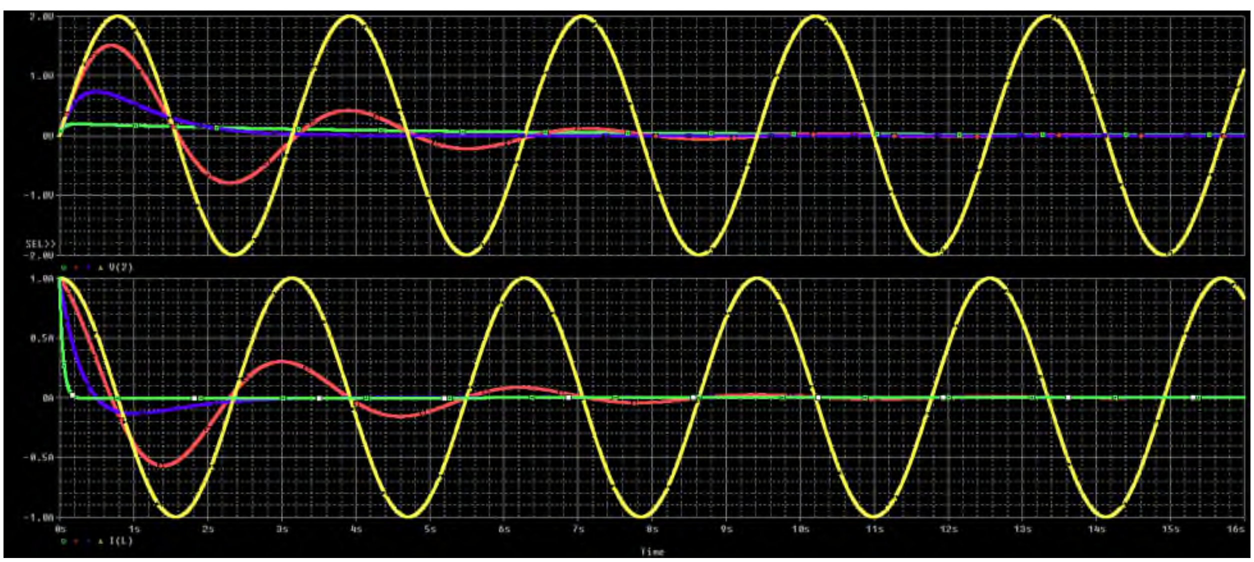
A comparison of the responses of the four parallel circuit examples (1.1, 2.1, 3.1 and 4.1) is shown below:

```
Example 1.1
C     1     0     {1/4}      IC=1
G     1     0     1     0    5
L     1     0     1     IC=0
.TRAN       1     16    0    1m    UIC
.PROBE
.END
Example 2.1
C     1     0     {1/4}      IC=1
G     1     0     1     0    {1/5}
L     1     0     1     IC=0
.TRAN       1     16    0    1m    UIC
.PROBE
.END
Example 3.1
C     1     0     {1/4}      IC=1
G     1     0     1     0    1
L     1     0     1     IC=0
.TRAN       1     16    0    1m    UIC
.PROBE
.END
Example 4.1
C     1     0     {1/4}      IC=1
L     1     0     1     IC=0
.TRAN       1     16    0    1m    UIC
.PROBE
.END
```



A comparison of the responses of the four series circuit examples (1.2, 2.2, 3.2 and 4.2) is shown below:

```
Example 1.2
L     0     1     1     IC=1
R     1     2     20
C     2     0     {1/4}       IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
Example 2.2
L     0     1     1     IC=1
R     1     2     {4/5}
C     2     0     {1/4}       IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
Example 3.2
L     0     1     1     IC=1
R     1     2     4
C     2     0     {1/4}       IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
Example 4.2
L     0     2     1     IC=1
C     2     0     {1/4}       IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
```



Systems with a Constant Input

Next consider systems with constant input, $z(t) = K$. In the case of electrical circuits, this means DC sources are applied. Equation (1.1) becomes:

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = A\omega_n^2 K \tag{1.137}$$

If we assume that the natural response of the system is exponential, then $y(t) = \beta e^{rt} + \lambda$, and

$$r^2 \beta e^{rt} + 2\zeta\omega_n r\beta e^{rt} + \omega_n^2\left(\beta e^{rt} + \lambda\right) = 0 \tag{1.138}$$

or

$$\left(r^2 + 2\zeta\omega_n r + \omega_n^2\right)\beta e^{rt} + \omega_n^2 \lambda = A\omega_n^2 K \tag{1.139}$$

which means that

$$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0 \tag{1.140}$$

$$\omega_n^2 \lambda = A\omega_n^2 K \quad \Rightarrow \quad \lambda = AK \tag{1.141}$$

Equation (1.140) is called the characteristic equation of the system, and it has roots given by:

$$\begin{aligned} r_{1,2} &= \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \end{aligned} \tag{1.142}$$

As in the unforced case, we will see that there are four distinctly different forms of the solution to equation (1.137), depending on the value of ζ with respect to the number 1.

Case 1

If $\zeta > 1$, then $\zeta^2 - 1 > 0$, and there will be two distinct negative real roots, $r_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$ and $r_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$. In this case, the system is said to be ***overdamped***, and because there are ***two*** roots to the characteristic equation, $y(t)$ will have ***two*** exponential components:

$$y(t) = \beta_1 e^{r_1 t} + \beta_2 e^{r_2 t} + AK \tag{1.143}$$

To determine the values of β_1 and β_2 note that

$$\dot{y}(t) = \beta_1 r_1 e^{r_1 t} + \beta_2 r_2 e^{r_1 t} \tag{1.144}$$

Evaluating equations (1.143) and (1.144) at $t = 0$, we have

$$\beta_1 + \beta_2 + AK = y(0) \tag{1.145}$$

and

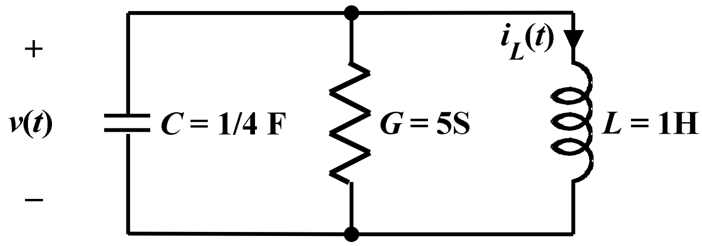
$$r_1\beta_1 + r_2\beta_2 = \dot{y}(0) \tag{1.146}$$

These two simultaneous equations can be used to evaluate β_1 and β_2 using Cramer's Rule as follows:

$$\beta_1 = \frac{\begin{vmatrix} y(0) - AK & 1 \\ \dot{y}(0) & r_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix}} = \frac{r_2\left[y(0) - AK\right] - \dot{y}(0)}{r_2 - r_1} \tag{1.147}$$

$$\beta_2 = \frac{\begin{vmatrix} 1 & y(0) - AK \\ r_1 & \dot{y}(0) \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix}} = \frac{\dot{y}(0) - r_1\left[y(0) - AK\right]}{r_2 - r_1} \tag{1.148}$$

Example 1.1



As shown by equation (1.7), this parallel RLC circuit can be described by the equation

$$\frac{d^2 i_L}{dt^2} + 20\frac{di_L}{dt} + 4i_L = 0 \tag{1.149}$$

Hence, the characteristic equation is

$$r^2 + 20r + 4 = 0 \tag{1.150}$$

and

$$\omega_n = \frac{1}{\sqrt{1(1/4)}} = 2 \tag{1.151}$$

$$\zeta = \frac{5}{2}\sqrt{\frac{1}{(1/4)}} = 5 \tag{1.152}$$

This is an overdamped system, with

$$r_1 = \left(-5 + \sqrt{25-1}\right)2 \approx -0.202 \tag{1.153}$$

$$r_2 = \left(-5 - \sqrt{25-1}\right)2 \approx -19.798 \tag{1.154}$$

Suppose now that $i_L(0) = 0$ and $v(0) = 1$. Then, $v(t) = L\frac{di_L(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{di_L}{dt}\right|_{t=0} = \frac{1}{L}v(0) = 1$, and

$$\beta_1 \approx \frac{(-19.798)(0) - 1}{-19.798 - (-0.202)} \approx \frac{-1}{-19.596} \approx 0.051 \tag{1.155}$$

$$\beta_2 \approx \frac{1 - (-0.202)(0)}{-19.798 - (-0.202)} \approx \frac{1}{-19.596} \approx -0.051 \tag{1.156}$$

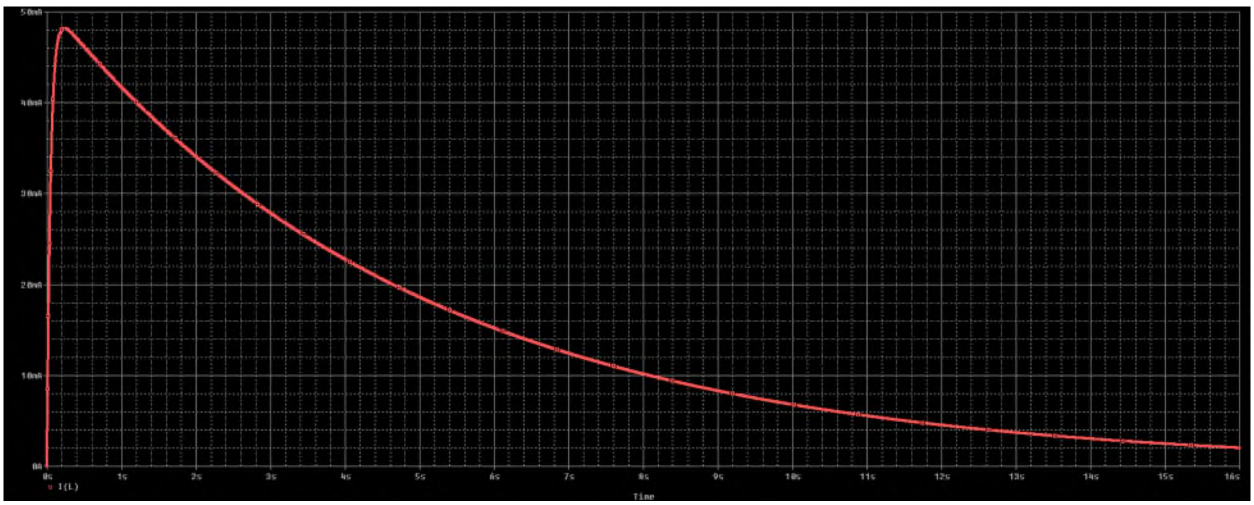
Hence,

$$i_L(t) \approx 0.051e^{-0.202t} - 0.051e^{-19.798t} \text{ A for } t > 0 \quad (1.157)$$

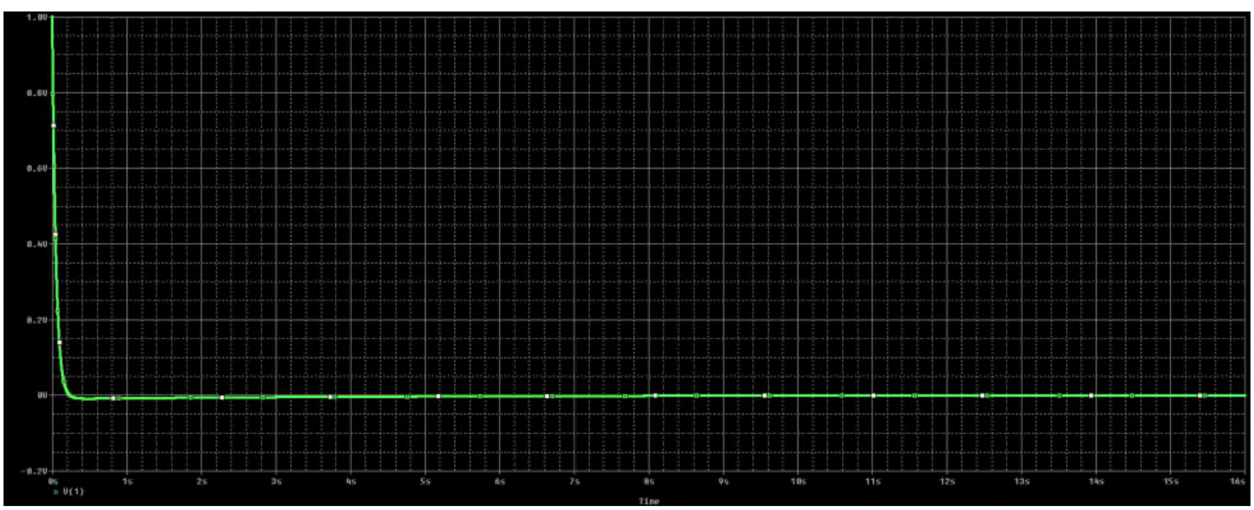
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 1.1
C    1    0    {1/4}     IC=1
G    1    0    1    0    5
L    1    0    1    IC=0
.TRAN     1    16   0    1m   UIC
.PROBE
.END
```

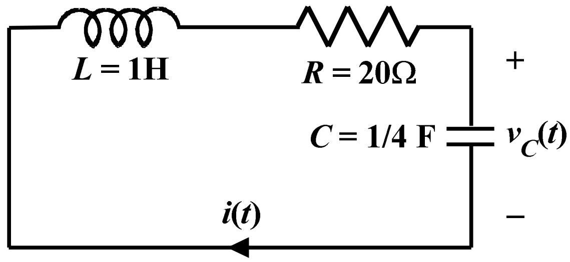
The inductor current is:



and the capacitor voltage is:



Example 1.2



As shown by equation (1.16), this series RLC circuit can be described by the equation

$$\frac{d^2 v_C}{dt^2} + 20\frac{dv_C}{dt} + 4v_C = 0 \tag{1.158}$$

Hence, the characteristic equation is

$$r^2 + 20r + 4 = 0 \tag{1.159}$$

and

$$\omega_n = \frac{1}{\sqrt{1(1/4)}} = 2 \tag{1.160}$$

$$\zeta = \frac{20}{2}\sqrt{\frac{(1/4)}{1}} = 5 \tag{1.161}$$

This is an overdamped system, with

$$r_1 = \left(-5 + \sqrt{25-1}\right)2 \approx -0.202 \tag{1.162}$$

$$r_2 = \left(-5 - \sqrt{25-1}\right)2 \approx -19.798 \tag{1.163}$$

Suppose now that $v_C(0) = 0$ and $i(0) = 1$. Then, $i(t) = C\frac{dv_C(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{dv_C(t)}{dt}\right|_{t=0} = \frac{1}{C}i(0) = 4$, and

$$\beta_1 \approx \frac{(-19.798)(0) - 4}{-19.798 - (-0.202)} \approx 0.204 \tag{1.164}$$

$$\beta_2 \approx \frac{4 - (-0.202)(0)}{-19.798 - (-0.202)} \approx -0.204 \tag{1.165}$$

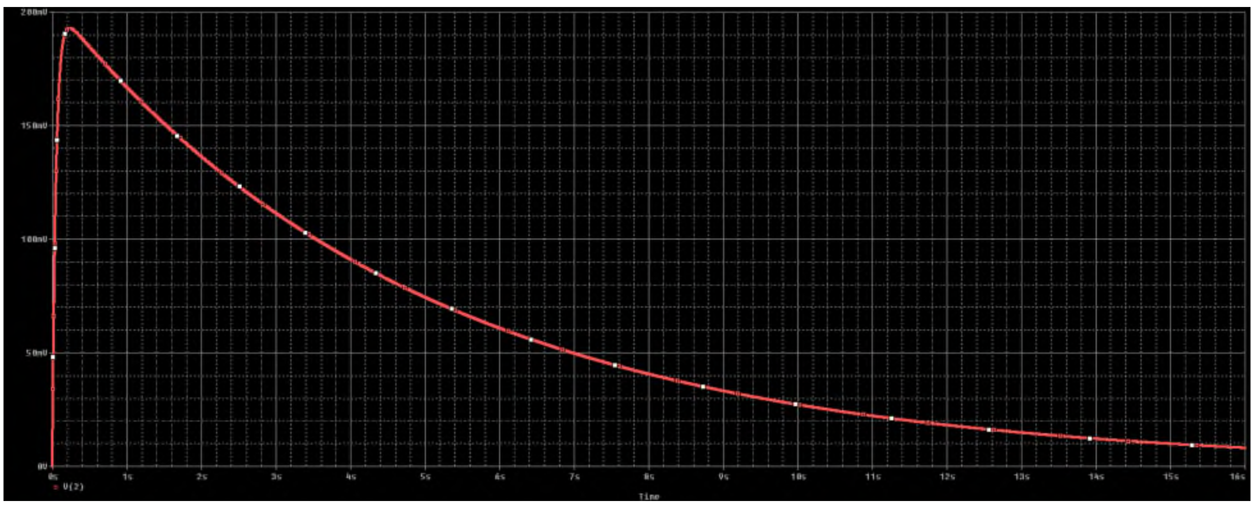
Hence,

$$v_C(t) \approx 0.204e^{-0.202t} - 0.204e^{-19.798t} \text{ V for } t > 0 \tag{1.166}$$

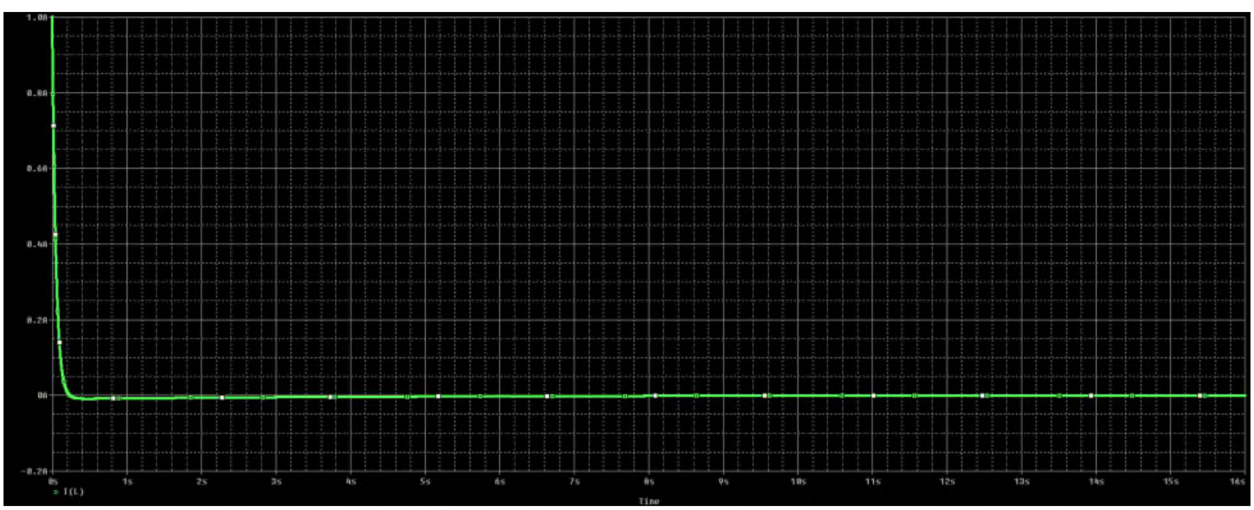
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 1.2
L     0    1    1    IC=1
R     1    2    20
C     2    0    {1/4}     IC=0
.TRAN      1    16   0    1m   UIC
.PROBE
.END
```

The capacitor voltage is:



and the inductor current is:



Case 2

If $0 < \zeta < 1$, then $\zeta^2 - 1 < 0$, and there will be two complex conjugate roots, $r_1 = \left(-\zeta + j\sqrt{1-\zeta^2}\right)\omega_n = -\zeta\omega_n + j\omega_d$ and $r_2 = \left(-\zeta - j\sqrt{1-\zeta^2}\right)\omega_n = -\zeta\omega_n - j\omega_d$. In this case, the system is said to be ***underdamped***., and the quantity $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is called the *damped* or *ringing* frequency.

As in Case 1, because there are ***two*** distinct roots to the characteristic equation, $y(t)$ has ***two*** exponential components:

$$\begin{aligned} y(t) &= \beta_1 e^{(-\zeta\omega_n + j\omega_d)t} + \beta_2 e^{(-\zeta\omega_n - j\omega_d)t} + AK \\ &= e^{-\zeta\omega_n t}\left(\beta_1 e^{j\omega_d t} + \beta_2 e^{-j\omega_d t}\right) + AK \end{aligned} \tag{1.167}$$

However, it is usually preferred to use Euler's identity

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta \tag{1.168}$$

to express $y(t)$ in the alternate form

$$\begin{aligned} y(t) &= e^{-\zeta\omega_n t}\left[\beta_1\left(\cos\omega_d t + j\sin\omega_d t\right) + \beta_2\left(\cos\omega_d t - j\sin\omega_d t\right)\right] + AK \\ &= e^{-\zeta\omega_n t}\left[\left(\beta_1 + \beta_2\right)\cos\omega_d t + j\left(\beta_1 - \beta_2\right)\sin\omega_d t\right] + AK \\ &= e^{-\zeta\omega_n t}\left[B_1\cos\omega_d t + B_2\sin\omega_d t\right] + AK \end{aligned} \tag{1.169}$$

where $B_1 = \beta_1 + \beta_2$ and $B_2 = j\left(\beta_1 - \beta_2\right)$.

To determine the values of B_1 and B_2 note that

$$\dot{y}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}\left[B_1\cos\omega_d t + B_2\sin\omega_d t\right] + e^{-\zeta\omega_n t}\left[-B_1\omega_d\sin\omega_d t + B_2\omega_d\cos\omega_d t\right] \tag{1.170}$$

Evaluating equations (1.169) and (1.170) at $t = 0$, we have

$$B_1 + AK = y(0) \tag{1.171}$$

and

$$-\zeta\omega_n B_1 + B_2\omega_d = \dot{y}(0) \tag{1.172}$$

Thus,

$$B_1 = y(0) - AK \tag{1.173}$$

and

$$B_2 = \frac{\dot{y}(0) + \zeta\omega_n B_1}{\omega_d} = \frac{\dot{y}(0) + \zeta\omega_n \left[y(0) - AK \right]}{\omega_d} \tag{1.174}$$

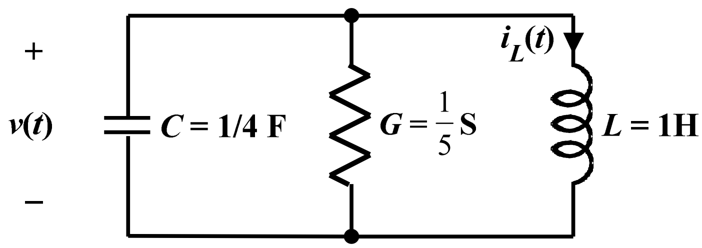
Alternately, note that

$$B_1 \cos \omega_d t + B_2 \sin \omega_d t = B_3 \cos\left(\omega_d t - \phi\right) \tag{1.175}$$

where $B_3 = \sqrt{B_1^2 + B_2^2}$ and $\phi = \tan^{-1}\left(\frac{B_2}{B_1}\right)$, so that $y(t)$ can be written in a slightly more compact form as

$$y(t) = B_3 e^{-\zeta\omega_n t} \cos\left(\omega_d t - \phi\right) + AK \tag{1.176}$$

Example 2.1



As shown by equation (1.7), this parallel RLC circuit can be described by the equation

$$\frac{d^2 i_L}{dt^2} + \frac{4}{5}\frac{di_L}{dt} + 4i_L = 0 \tag{1.177}$$

Hence, the characteristic equation is

$$r^2 + \frac{4}{5}r + 4 = 0 \tag{1.178}$$

and

$$\omega_n = 2 \tag{1.179}$$

$$\zeta = \frac{1}{10}\sqrt{4} = 0.2 \tag{1.180}$$

$$\omega_d = 2\sqrt{1-(0.2)^2} \approx 1.960 \tag{1.181}$$

This is an underdamped system, with

$$r_{1,2} \approx -0.400 + j1.960 \tag{1.182}$$

$$r_{1,2} \approx -0.400 - j1.960 \tag{1.183}$$

Suppose now that $i_L(0) = 0$ and $v(0) = 1$. Then, $v(t) = L\frac{di_L(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{di_L(t)}{dt}\right|_{t=0} = \frac{1}{L}v(0) = 1$, and

$$B_1 = 0 \tag{1.184}$$

$$B_2 \approx \frac{1+(0.2)(2)(0)}{1.960} \approx 0.510 \tag{1.185}$$

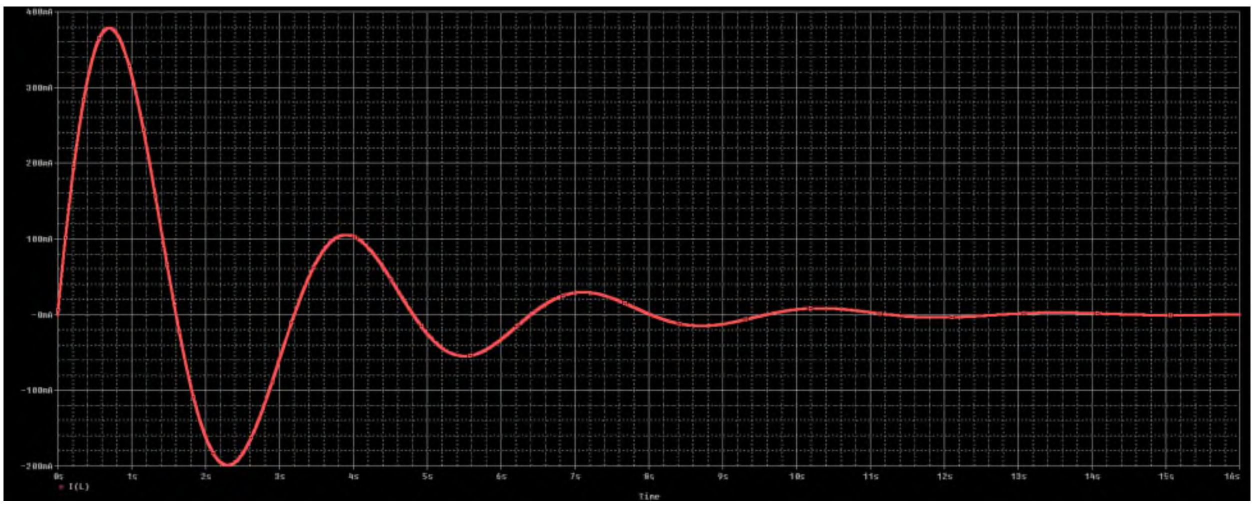
Hence,

$$i_L(t) \approx 0.510e^{-0.4t}\sin(1.960t) \text{ A for } t > 0 \quad (1.186)$$

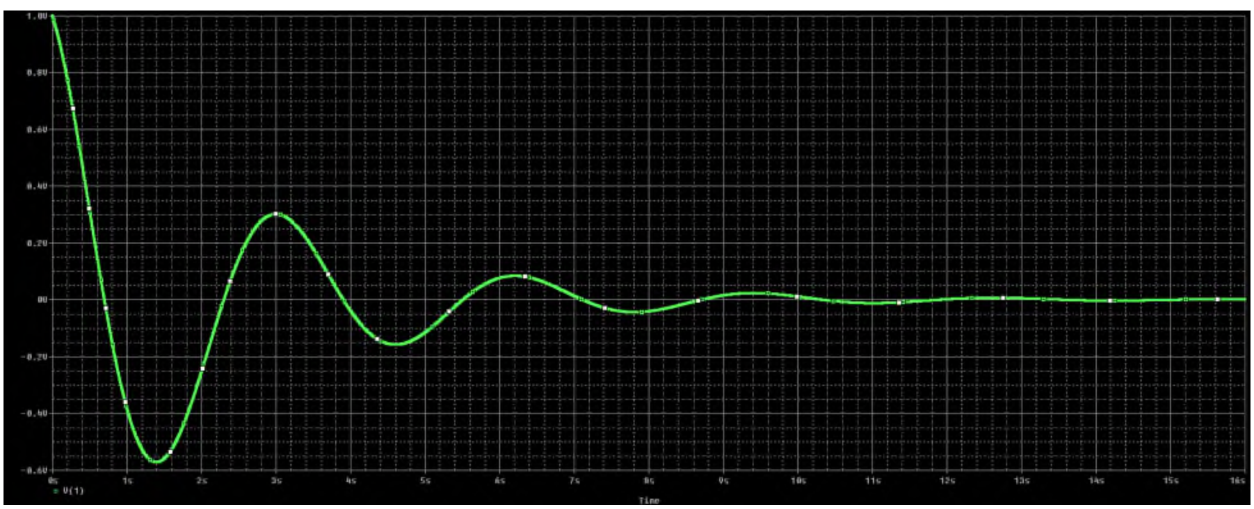
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 2.1
C     1    0    {1/4}     IC=1
G     1    0    1    0    {1/5}
L     1    0    1    IC=0
.TRAN      1    16   0    1m   UIC
.PROBE
.END
```

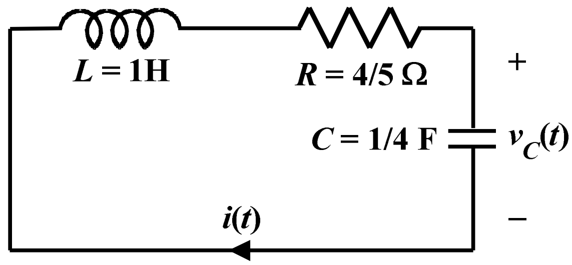
The inductor current is:



and the capacitor voltage is:



Example 2.2



As shown by equation (1.16), this series RLC circuit can be described by the equation

$$\frac{d^2 v_C}{dt^2} + \frac{4}{5}\frac{dv_C}{dt} + 4v_C = 0 \tag{1.187}$$

Hence, the characteristic equation is

$$r^2 + \frac{4}{5}r + 4 = 0 \tag{1.188}$$

and

$$\omega_n = 2 \tag{1.189}$$

$$\zeta = \frac{2}{5}\sqrt{\frac{1}{4}} = 0.2 \tag{1.190}$$

$$\omega_d = 2\sqrt{1-(0.2)^2} \approx 1.960 \tag{1.191}$$

This is an underdamped system, with

$$r_1 = -0.400 + j1.960 \tag{1.192}$$

$$r_2 = -0.400 - j1.960 \tag{1.193}$$

Suppose now that $v_C(0) = 0$ and $i(0) = 1$. Then, $i(t) = C\frac{dv_C(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{dv_C(t)}{dt}\right|_{t=0} = \frac{1}{C}i(0) = 4$, and

$$B_1 = 0 \tag{1.194}$$

$$B_2 \approx \frac{4+(0.2)(2)(0)}{1.960} \approx 2.041 \quad (1.195)$$

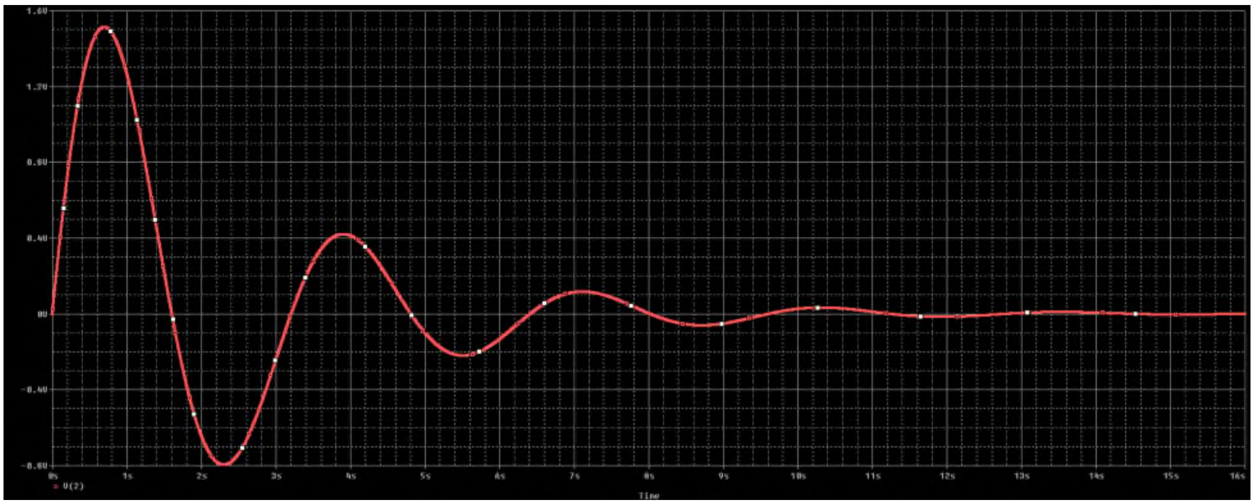
Hence,

$$v_C(t) = 2.041e^{-0.4t}\sin(1.960t) \;\; \text{V} \quad t > 0 \quad (1.196)$$

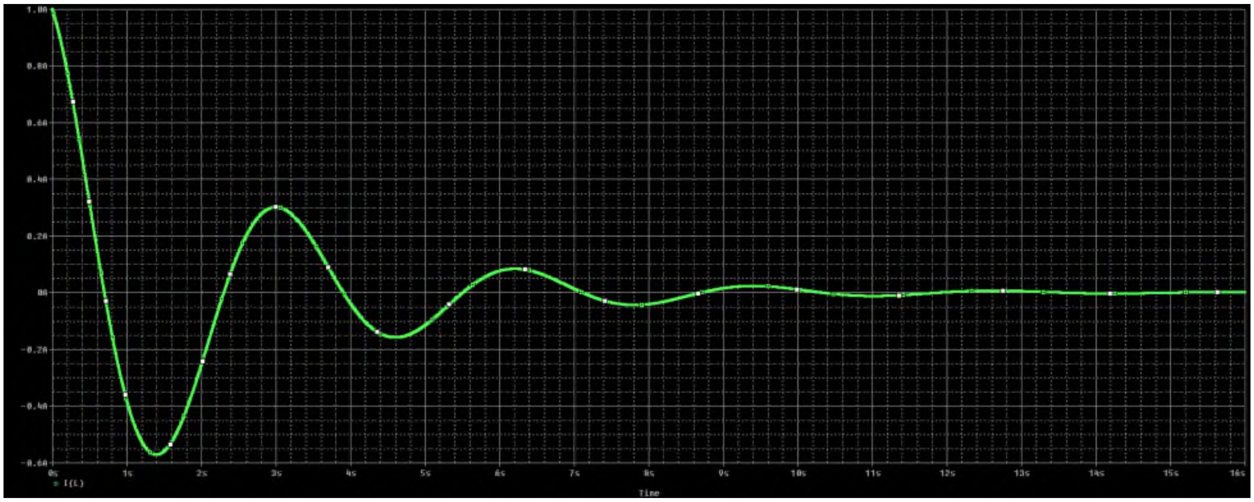
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 2.2
L     0     1     1     IC=1
R     1     2     {4/5}
C     2     0     {1/4}       IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
```

The capacitor voltage is:



and the inductor current is:


1.0A
0.8A
0.6A
0.4A
0.2A
0A
-0.2A
-0.4A
-0.6A
0s
1s
2s
3s
4s
5s
6s
7s
8s
9s
10s
11s
12s
13s
14s
15s
16s
I(L)
Time

Case 3

If $\zeta = 1$, then $\zeta^2 - 1 = 0$, and there will be two identical negative real roots, $r_1 = r_2 = -\omega_n$. In this case, the system is said to be ***critically damped***. This case can be considered to be the "borderline" between overdamped and underdamped systems.

The general form of the solution is

$$y(t) = (\beta_1 + \beta_2 t)e^{-\omega_n t} + AK \tag{1.197}$$

To determine the values of β_1 and β_2 note that

$$\dot{y}(t) = \beta_2 e^{-\omega_n t} - \omega_n (\beta_1 + \beta_2 t)e^{-\omega_n t} \tag{1.198}$$

Evaluating equations (1.197) and (1.198) at $t = 0$, we have

$$\beta_1 + AK = y(0) \tag{1.199}$$

and

$$\beta_2 - \omega_n \beta_1 = \dot{y}(0) \tag{1.200}$$

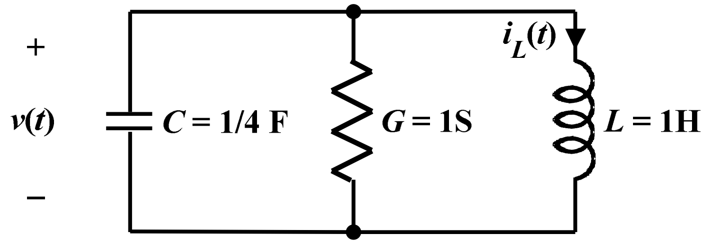
Thus,

$$\beta_1 = y(0) - AK \tag{1.201}$$

and

$$\beta_2 = \dot{y}(0) + \omega_n \beta_1 = \dot{y}(0) + \omega_n \left[y(0) - AK \right] \tag{1.202}$$

Example 3.1



As shown by equation (1.7), this parallel circuit can be described by the equation

$$\frac{d^2 i_L}{dt^2} + 4\frac{di_L}{dt} + 4i_L = 0 \tag{1.203}$$

Hence, the characteristic equation is

$$r^2 + 4r + 4 = 0 \tag{1.204}$$

and

$$\omega_n = 2 \tag{1.205}$$

$$\zeta = \frac{1}{2}\sqrt{4} = 1 \tag{1.206}$$

This is a critically damped system, with

$$r_1 = r_2 = -2 \tag{1.207}$$

Suppose now that $i_L(0) = 0$ and $v(0) = 1$. Then, $v(t) = L\frac{di_L(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{di_L}{dt}\right|_{t=0} = \frac{1}{L}v(0) = 1$, and

$$\beta_1 = 0 \tag{1.208}$$

$$\beta_2 = 1 + (2)(0) = 1 \tag{1.209}$$

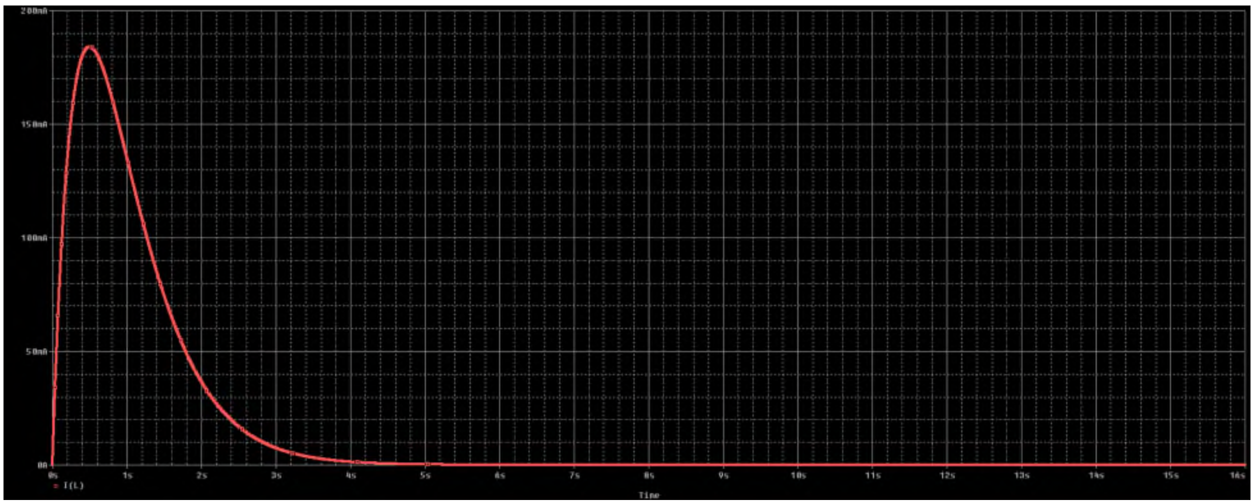
Hence,

$$i_L(t) = te^{-2t} \text{ A for } t > 0 \tag{1.210}$$

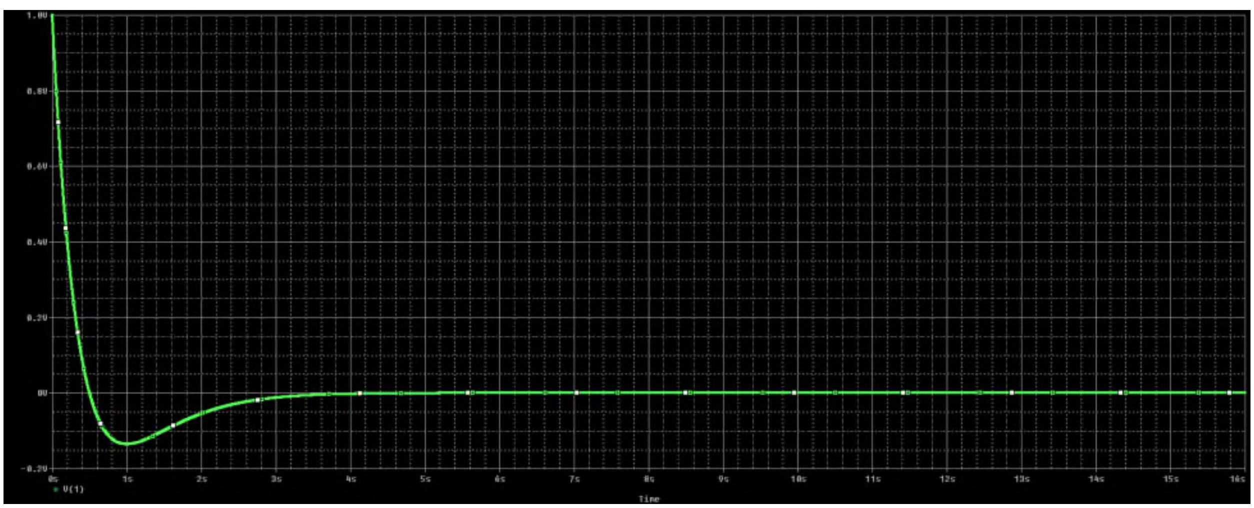
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 3.1
C     1     0     {1/4}      IC=1
G     1     0     1     0     1
L     1     0     1     IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
```

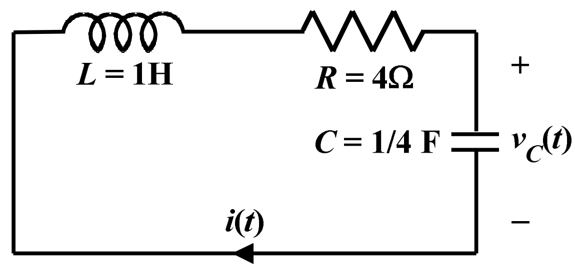
The inductor current is:



and the capacitor voltage is:



Example 3.2



As shown by equation (1.16), this series RLC circuit can be described by the equation

$$\frac{d^2v_C}{dt^2}+4\frac{dv_C}{dt}+4v_C=0 \tag{1.211}$$

Hence, the characteristic equation is

$$r^2+4r+4=0 \tag{1.212}$$

and

$$\omega_n=2 \tag{1.213}$$

$$\zeta=\frac{4}{2}\sqrt{\frac{1}{4}}=1 \tag{1.214}$$

This is a critically damped system, with

$$r_1=r_2=-2 \tag{1.215}$$

Suppose now that $v_C(0)=0$ and $i(0)=1$. Then, $i(t)=C\frac{dv_C(t)}{dt}$, when evaluated at $t=0$, yields $\left.\frac{dv_C(t)}{dt}\right|_{t=0}=\frac{1}{C}i(0)=4$, and

$$\beta_1=0 \tag{1.216}$$

$$\beta_2=4+(2)(0)=4 \tag{1.217}$$

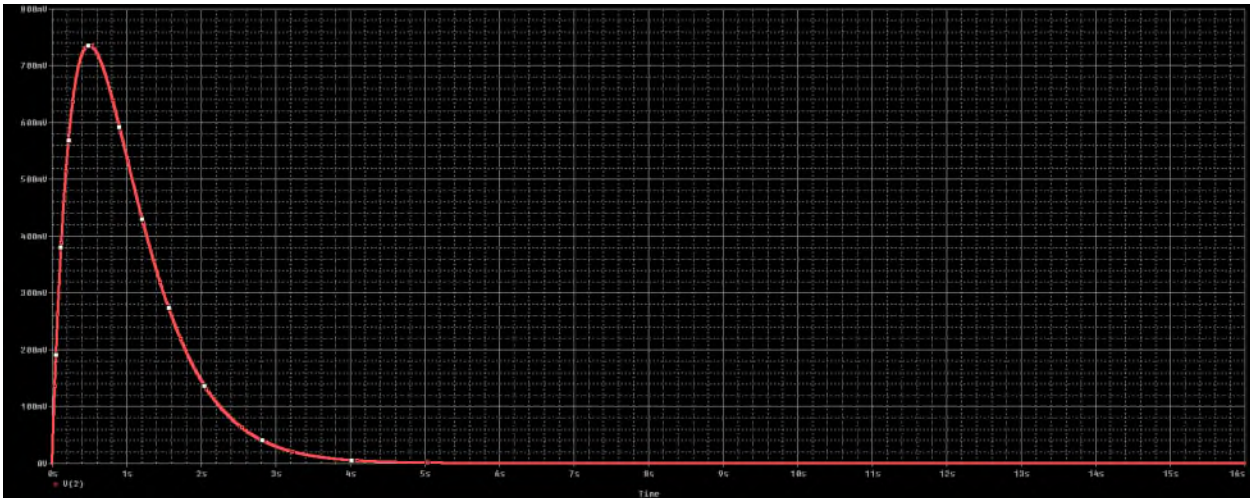
Hence,

$$v_C(t)=4te^{-2t} \text{ V} \quad t>0 \tag{1.218}$$

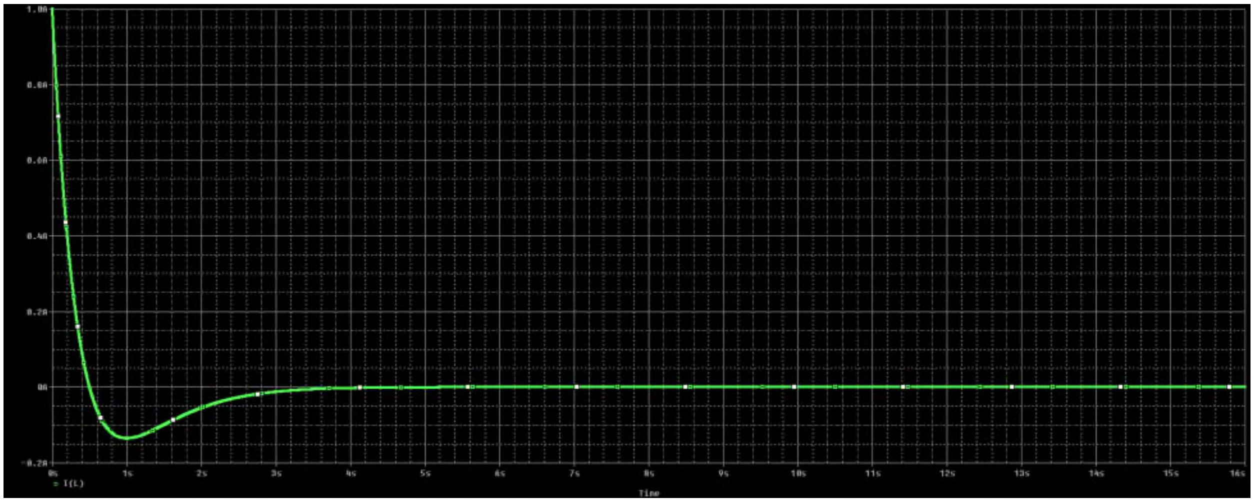
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 3.2
L     0    1    1     IC=1
R     1    2    4
C     2    0    {1/4}      IC=0
.TRAN      1    16   0     1m    UIC
.PROBE
.END
```

The capacitor voltage is:



and the inductor current is:



Case 4

If $\zeta = 0$, then $\zeta^2 - 1 = -1$, and there will be two conjugate imaginary roots, $r_{1,2} = \pm j\omega_n$. In this case, the system is said to be ***undamped***.

As there are ***two*** distinct roots to the characteristic equation, $y(t)$ has ***two*** exponential components

$$y(t) = \beta_1 e^{j\omega_n t} + \beta_2 e^{-j\omega_n t} + AK \tag{1.219}$$

Here again, as in Case 2, it is usually preferred to use Euler's identity to express $y(t)$ in the alternate form

$$\begin{aligned} y(t) &= \beta_1 (\cos\omega_n t + j\sin\omega_n t) + \beta_2 (\cos\omega_n t - j\sin\omega_n t) + AK \\ &= (\beta_1 + \beta_2)\cos\omega_n t + j(\beta_1 - \beta_2)\sin\omega_n t + AK \\ &= B_1 \cos\omega_n t + B_2 \sin\omega_n t + AK \end{aligned} \tag{1.220}$$

where $B_1 = \beta_1 + \beta_2$ and $B_2 = j(B_1 - B_2)$.

To determine the values of B_1 and B_2 note that

$$\dot{y}(t) = -\omega_n B_1 \sin\omega_n t + \omega_n B_2 \cos\omega_n t \tag{1.221}$$

Evaluating equations (1.220) and (1.221) at $t = 0$, we have

$$B_1 + AK = y(0) \tag{1.222}$$

and

$$\omega_n B_2 = \dot{y}(0) \tag{1.223}$$

so that

$$B_1 = y(0) - AK \tag{1.224}$$

and

$$B_2 = \frac{\dot{y}(0)}{\omega_n} \tag{1.225}$$

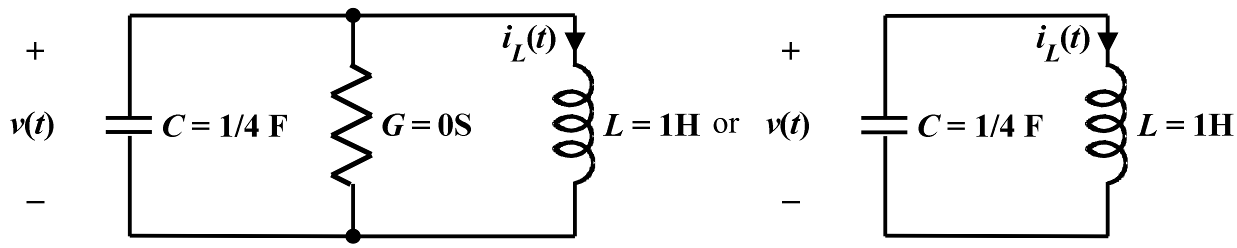
Alternately, note that

$$B_1 \cos\omega_n t + B_2 \sin\omega_n t = B_3 \cos(\omega_n t - \phi) \tag{1.226}$$

where $B_3 = \sqrt{B_1^2 + B_2^2}$ and $\phi = \tan^{-1}\left(\frac{B_2}{B_1}\right)$, so that $y(t)$ can be written in a slightly more compact form as

$$y(t) = B_3 \cos(\omega_n t - \phi) + AK \quad (1.227)$$

Example 4.1



As shown by equation (1.7), this parallel RLC circuit can be described by the equation

$$\frac{d^2 i_L}{dt^2} + 4 i_L = 0 \tag{1.228}$$

Hence, the characteristic equation is

$$r^2 + 4 = 0 \tag{1.229}$$

and

$$\omega_n = 2 \tag{1.230}$$

$$\zeta = 0 \tag{1.231}$$

This is an undamped system, with

$$r_1 = j2 \tag{1.232}$$

$$r_2 = -j2 \tag{1.233}$$

Suppose now that $i_L(0) = 0$ and $v(0) = 1$. Then, $v(t) = L\frac{di_L(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{di_L}{dt}\right|_{t=0} = \frac{1}{L}v(0) = 1$, and

$$B_1 = 0 \tag{1.234}$$

$$B_2 = \frac{1}{2} \tag{1.235}$$

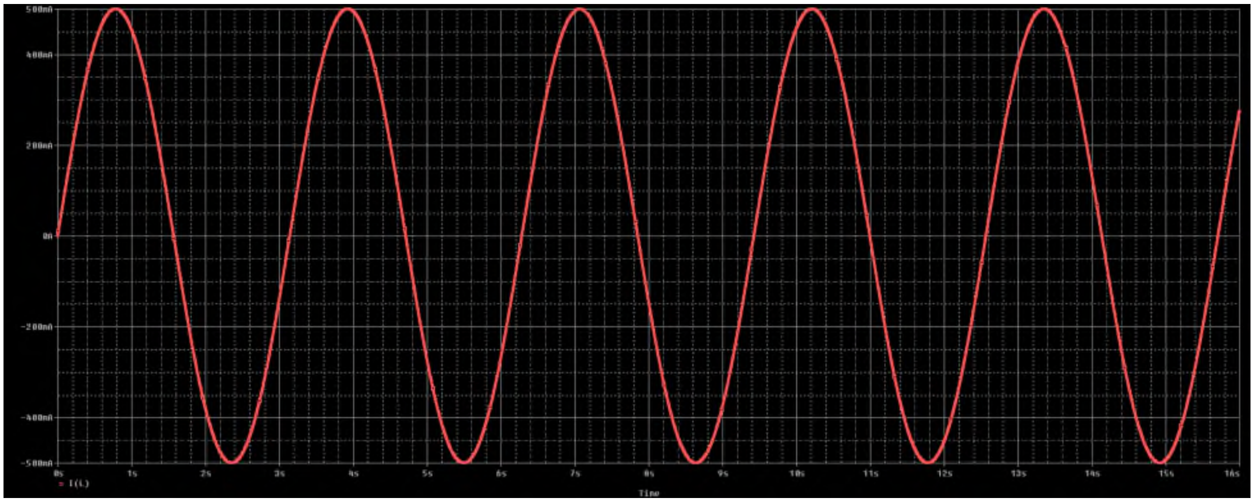
Hence,

$$i_L(t) = \frac{1}{2}\sin 2t \ \text{ A for } \ t > 0 \tag{1.236}$$

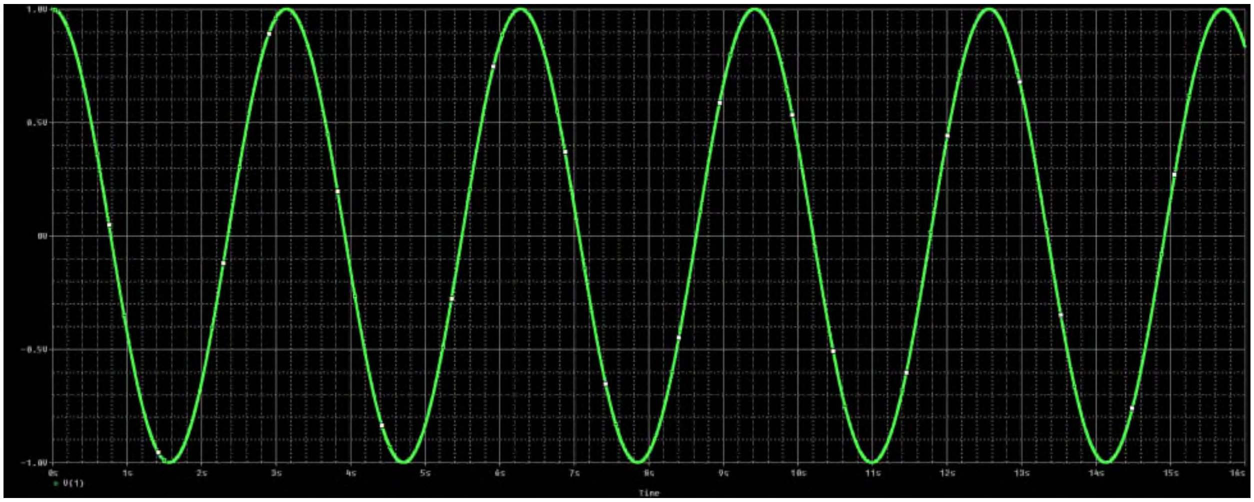
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 4.1
C     1     0     {1/4}     IC=1
L     1     0     1     IC=0
.TRAN     1     16     0     1m     UIC
.PROBE
.END
```

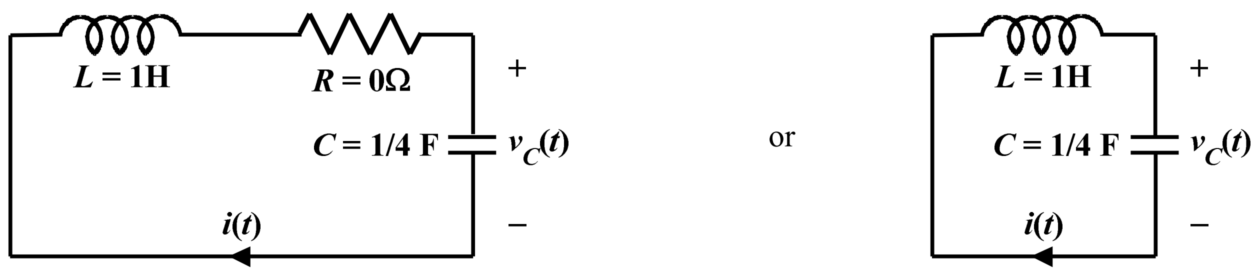
The inductor current is:



and the capacitor voltage is:



Example 4.2



As shown by equation (1.16), this series RLC circuit can be described by the equation

$$\frac{d^2 v_C}{dt^2} + 4v_C = 0 \tag{1.237}$$

Hence, the characteristic equation is

$$r^2 + 4 = 0 \tag{1.238}$$

and

$$\omega_n = 2 \tag{1.239}$$

$$\zeta = 0 \tag{1.240}$$

This is an undamped system, with

$$r_1 = j2 \tag{1.241}$$

$$r_2 = -j2 \tag{1.242}$$

Suppose now that $v_C(0) = 0$ and $i(0) = 1$. Then, $i(t) = C\frac{dv_C(t)}{dt}$, when evaluated at $t = 0$, yields $\left.\frac{dv_C(t)}{dt}\right|_{t=0} = \frac{1}{C}i(0) = 4$, and

$$B_1 = 0 \tag{1.243}$$

$$B_2 = \frac{4}{2} = 2 \tag{1.244}$$

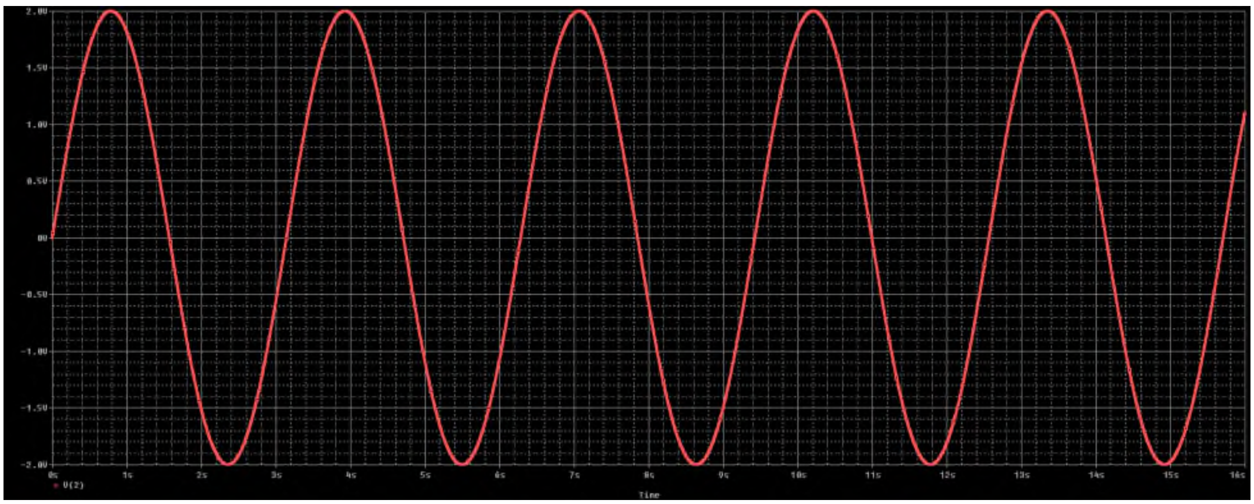
Hence,

$$v_C(t) = 2\sin 2t \quad \text{V} \quad t > 0 \tag{1.245}$$

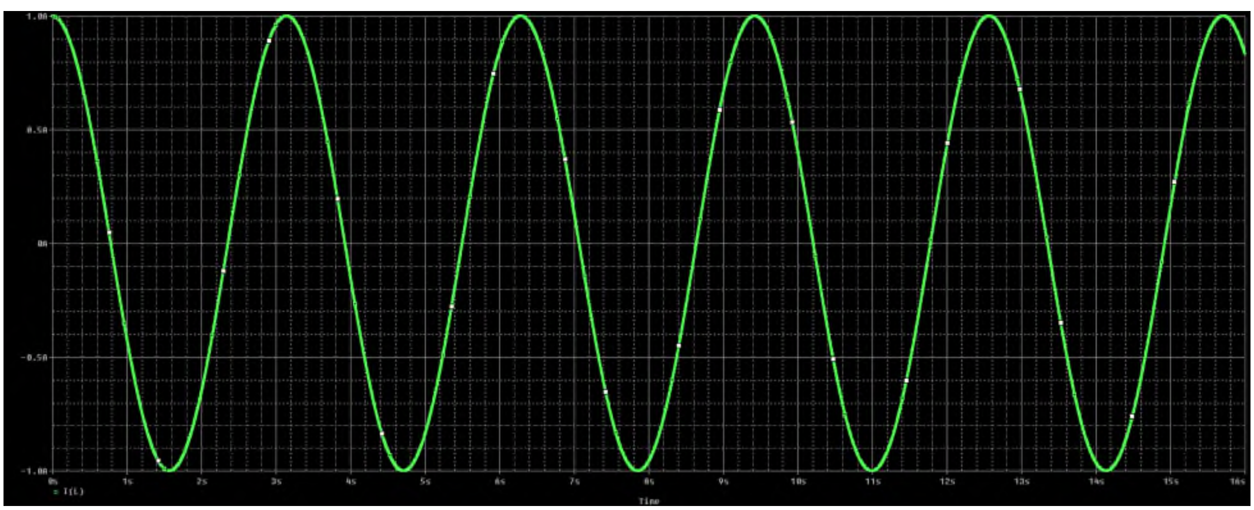
To see what this looks like, we can simulate the circuit with PSpice as follows:

```
Example 4.2
L     0     2     1     IC=1
C     2     0     {1/4}      IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
```

The capacitor voltage is:

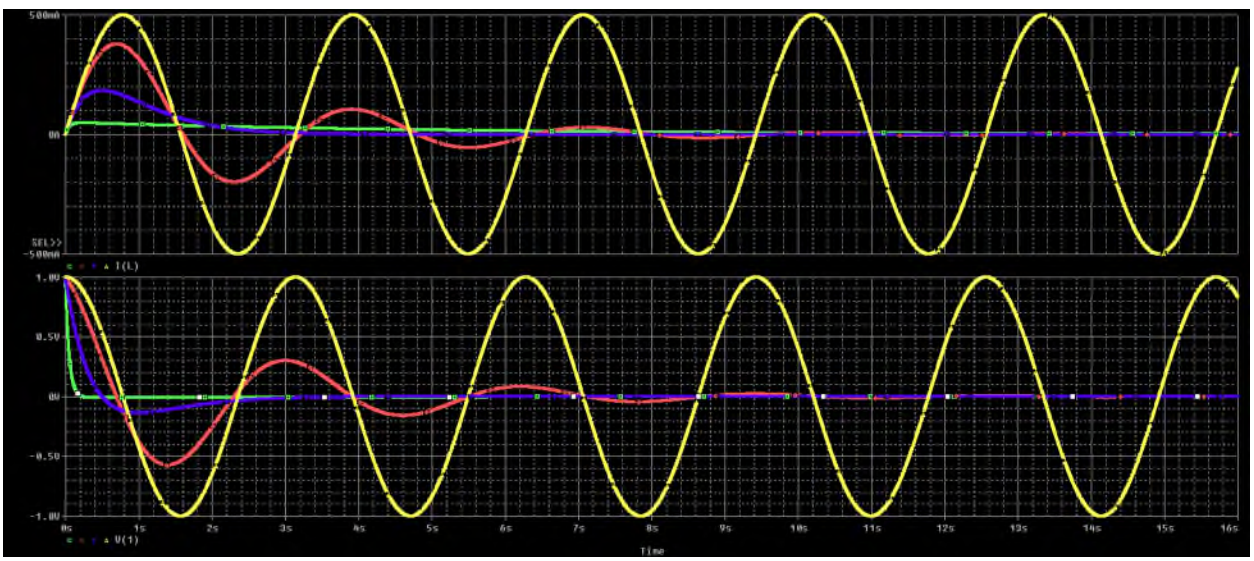


and the inductor current is:



A comparison of the responses of the four parallel circuit examples (1.1, 2.1, 3.1 and 4.1) is shown below:

```
Example 1.1
C     1     0     {1/4}     IC=1
G     1     0     1     0     5
L     1     0     1     IC=0
.TRAN      1     16    0     1m    UIC
.PROBE
.END
Example 2.1
C     1     0     {1/4}     IC=1
G     1     0     1     0     {1/5}
L     1     0     1     IC=0
.TRAN      1     16    0     1m    UIC
.PROBE
.END
Example 3.1
C     1     0     {1/4}     IC=1
G     1     0     1     0     1
L     1     0     1     IC=0
.TRAN      1     16    0     1m    UIC
.PROBE
.END
Example 4.1
C     1     0     {1/4}     IC=1
L     1     0     1     IC=0
.TRAN      1     16    0     1m    UIC
.PROBE
.END
```



A comparison of the responses of the four series circuit examples (1.2, 2.2, 3.2 and 4.2) is shown below:

```
Example 1.2
L     0     1     1     IC=1
R     1     2     20
C     2     0     {1/4}       IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
Example 2.2
L     0     1     1     IC=1
R     1     2     {4/5}
C     2     0     {1/4}       IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
Example 3.2
L     0     1     1     IC=1
R     1     2     4
C     2     0     {1/4}       IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
Example 4.2
L     0     2     1     IC=1
C     2     0     {1/4}       IC=0
.TRAN       1     16    0     1m    UIC
.PROBE
.END
```